Percolation and Random walks on graphs

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Contents

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1 Percolation

These notes are largely based on Grimmett's books Percolation [3] and Probability on Graphs [4].

1.1 Definition of the model

Let $G = (V, E)$ be a graph with set of vertices V and set of edges E and let $p \in [0, 1]$. There are two models of percolation, bond and site percolation. We will mainly be interested in bond percolation. To define it consider the probability space $\Omega = \{0, 1\}^E$ and endow it with the σ-algebra generated by the cylinder sets, i.e. sets of the form

$$\{\omega(e) = x_e, \ \forall\, e \in A\},$$

where A is a finite set and $x_e \in \{0, 1\}$ for all e. The associated probability measure is going to be the product measure $\mathbb{P}_p(\cdot)$, which means that each edge is 1 (in this case we call it open) with probability p and 0 (closed) with probability $1-p$ and the states of different edges are independent. We will denote the state of the system by a vector $\eta_p \in \{0, 1\}^E$.

For this course we will focus on the case where $G = \mathbb{L}^d$, i.e. the d-dimensional Euclidean lattice.

We write $x \leftrightarrow y$ if there is an open path of edges from x to y. We write

$$\mathcal{C}(x) = \{y : y \leftrightarrow x\}$$

to be the cluster of x. We also write $x \leftrightarrow \infty$ if $|\mathcal{C}(x)| = \infty$. By translation invariance, it is clear that $|\mathcal{C}(x)|$ has the same distribution as $|\mathcal{C}(0)|$ for all x. We can now ask the following questions:

- Is there $p \in (0, 1)$ such that $\mathbb{P}_p(|\mathcal{C}(0)| = \infty) > 0$?
- Is the probability $\mathbb{P}_p(|\mathcal{C}(0)| = \infty)$ monotone in p?
- In the case where $\mathbb{P}_p(|\mathcal{C}(0)| = \infty) > 0$, how many infinite clusters are there?

To answer the questions above and more, we will introduce some machinery. The first tool that we will use a lot to study percolation is a coupling of all percolation processes.

Definition 1.1. Coupling of two probability measures μ and ν, defined on possibly different probability spaces, is a random variable (X, Y) defined on a single probability space so that the marginal distribution of X is μ and the marginal of Y is ν.

We will see the importance of coupling a lot of times during this course. From the definition it's not clear why this is such an important notion. If you have taken a Markov chains course before, you must have seen the use of coupling when you proved the convergence to stationarity.

1.2 Coupling of percolation processes

We are going to couple all percolation processes on $\mathbb{Z}^d$ for different values of p. To emphasise the dependence on p we write $\eta_p(e)$ to denote the state of the edge e when percolation probability is p.

Let $(U(e))_{e\in\mathbb{L}^d}$ be i.i.d. uniform random variables on $[0,1]$. We set

$$\eta_p(e) = \begin{cases} 1 & \text{if } U(e) \leq p \\ 0 & \text{otherwise.} \end{cases}$$

Then η_p has the law of bond percolation with parameter p.

Using this coupling we now immediately see that if $p \leq q$, then $\eta_p(e) \leq \eta_q(e)$, and hence this proves

Lemma 1.2. *The probability $\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty)$ is increasing as a function of p.*

1.3 Phase transition

We now define

$$p_c(d) = \sup\{p \in [0,1] : \theta(p) = 0\}$$

and using the monotonicity of the function $\theta(p)$ we get that if $p > p_c$, then $\theta(p) > 0$, while for $p < p_c$ we have $\theta(p) = 0$.

It is known that $\theta(p)$ is a C^∞ function on $(p_c, 1]$, but it is not known whether $\theta(p)$ is continuous at p_c for $d = 3$.

Conjecture 1.3. *The function $\theta(p)$ is continuous at p_c for $d = 3$.*

We next ask: is $p_c \in (0,1)$? We will answer this in the next theorem.

When $d = 1$, it is obvious that for all $p < 1$ we have $\mathbb{P}_p(|\mathcal{C}(0)| = \infty) = 0$, since with probability one we will encounter closed edges. Therefore, for $d = 1$ we have $p_c = 1$. The picture is different for higher dimensions.

Theorem 1.4. *For all $d \geq 2$ we have $p_c(d) \in (0,1)$.*

Proof that $p_c(d) > 0$. Let Σ_n denote the number of open self-avoiding paths of length n starting from 0. We clearly have

$$\mathbb{P}_p(|\mathcal{C}(0)| = \infty) = \mathbb{P}_p(\forall\, n : \Sigma_n \geq 1) = \lim_{n\to\infty} \mathbb{P}_p(\Sigma_n \geq 1) \leq \lim_{n\to\infty} \mathbb{E}_p[\Sigma_n]\,,$$

where for the last inequality we used Markov's inequality. We now need to upper bound $\mathbb{E}_p[\Sigma_n]$. Let σ_n denote the number of self-avoiding paths starting from 0 of length n. Then

$$\mathbb{E}_p[\Sigma_n] \leq \sigma_n \cdot p^n,$$

so it suffices to upper bound σ_n. A crude upper bound is as follows: for the first step we have $2d$ possible choices. For later steps, there are at most $2d - 1$ choices. Therefore

$$\mathbb{E}_p[\Sigma_n] \leq \sigma_n \cdot p^n \leq (2d) \cdot (2d-1)^{n-1} p^n = \frac{2d}{2d-1} \cdot ((2d-1)p)^n.$$

If $(2d-1)p < 1$, then $\lim_{n\to\infty} \mathbb{E}_p[\Sigma_n] = 0$. Therefore, this shows that $p_c(d) \geq \frac{1}{2d-1}$. $\square$

1.3.1 Self-avoiding walks

In the proof above we used the number of self-avoiding walks (paths that do not contain circuits) and used a very crude bound on their number. Let's see what else we can say about them.

Lemma 1.5. *Let σ_n be the number of self-avoiding paths of length n. Then for all m, n we have*

$$\sigma_{n+m} \leq \sigma_n \cdot \sigma_m.$$

Proof. The proof of the result above follows from the observation that a self-avoiding path of length $n+m$ can be written uniquely as a concatenation of a self-avoiding path of length n started from 0 and a translation of another self-avoiding path of length m. □

Corollary 1.6. *There is a constant λ so that*

$$\lim_{n\to\infty} \frac{\log \sigma_n}{n} = \lambda.$$

Proof. From the previous lemma, the sequence $\log \sigma_n$ is subadditive, i.e.

$$\log \sigma_{n+m} \leq \log \sigma_n + \log \sigma_m.$$

Exercise 1 completes the proof. □

Remark 1.7. From the above corollary we get that the number of self-avoiding paths of length n grows like $\sigma_n = e^{n\lambda(1+o(1))}$. We now define $\kappa = e^{\lambda}$ to be the connective constant of the lattice.

There are many open questions in this area. For instance, what is κ for the Euclidean lattice? It has been established recently by Duminil-Copin and Smirnov that $\kappa = \sqrt{2+\sqrt{2}}$ for the hexagonal lattice.

Conjecture 1.8. *The number of self-avoiding walks satisfies*

$$\sigma_n \approx \begin{cases} n^{\frac{11}{32}}\kappa^n & d = 2 \\ n^{\gamma}\kappa^n & d = 3 \\ (\log n)^{1/4}\kappa^n & d = 4 \\ \kappa^n & d \geq 5. \end{cases}$$

This conjecture was verified for $d \geq 5$ in the work of Hara and Slade using lace expansion.

For lower d, in particular $d = 2$, the bounds are still very far from the truth.

Theorem 1.9 (Hammersley and Welsh)**.** *For all d the number of self avoiding walks σ_n satisfies*

$$\sigma_n \leq \exp\left(c_d\sqrt{n}\right)\kappa_d^n,$$

where c_d is a positive constant.

Tom Hutchcroft proved a small improvement last month:

Theorem 1.10 (Hutchcroft)**.** *For all d we have*

$$\sigma_n \leq \exp(o(\sqrt{n}))\kappa^n.$$

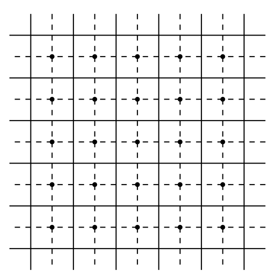
The study of the self-avoiding walk is a central topic in probability theory. Another big open question is to understand the scaling limit of a self-avoiding walk of length n started from the origin sampled uniformly at random. It is believed that this is characterised by an SLE(8/3) process, but there is no proof. Last summer, Ewain Gwynne and Jason Miller proved this statement for self-avoiding walks on random surfaces.

1.3.2 Existence and uniqueness of the infinite cluster

Definition 1.11. Let G be a planar graph, i.e. a graph that can be embedded on the plane in such a way that no edges cross.

The **dual** of a planar graph G (called primal) is obtained by placing a vertex in each face of G and connecting two vertices by an edge if their faces share a boundary edge.

Remark 1.12. The lattice $\mathbb{Z}^2$ satisfies the **duality property**, which means that $\mathbb{Z}^2$ is isomorphic to its dual lattice.



Proof of Theorem 1.4. ($p_c(d) < 1$) First of all we note that it is enough to prove that $p_c(2) < 1$, since for all d we can embed $\mathbb{Z}^d$ into $\mathbb{Z}^{d+1}$ and hence, if there exists an infinite cluster in $\mathbb{Z}^d$, this would imply the same statement for $\mathbb{Z}^{d+1}$.

The reason why we choose to prove for $d = 2$ is that the 2-dimensional lattice satisfies the duality property, Remark 1.12.

The following argument is usually referred to as *Peierls argument*. (Peierls used it to prove phase transition in the two-dimensional Ising model.) Consider the dual of $\mathbb{Z}^2$ and perform bond percolation, by declaring an edge open if the edge that it crosses from $\mathbb{Z}^2$ is open and vice versa. It is clear from a picture, that $|\mathcal{C}(0)| < \infty$ if and only if there exists a closed dual circuit that contains 0. Let D_n denote the number of closed dual circuits of length n that surround the origin. Then we obtain

$$\mathbb{P}_p(|\mathcal{C}(0)| < \infty) = \mathbb{P}_p(\exists\, n:\ D_n \geq 1) \leq \sum_{n=4}^{\infty} \mathbb{E}_p[D_n]\,.$$

Exercise 3 shows that the number of dual circuits of length n that surround 0 is at most $n4^n$. So we obtain

$$\mathbb{E}_p[D_n] \leq n4^n(1-p)^n.$$

So by taking p sufficiently close to 1, say $p > 1-\delta$, we can make $\mathbb{P}_p(|\mathcal{C}(0)| < \infty) < 1-\varepsilon$, which means that $p_c(d) \le 1-\delta$. $\square$

Lemma 1.13. *Let A_∞ be the event that there exists an infinite cluster. Then we have the following dichotomy:*

(a) *If $\theta(p) = 0$, then $\mathbb{P}_p(A_\infty) = 0$.*

(b) *If $\theta(p) > 0$, then $\mathbb{P}_p(A_\infty) = 1$.*

Proof. (a) Suppose that $\theta(p) = 0$. By the union bound we have

$$\mathbb{P}_p(A_\infty) = \mathbb{P}_p(\exists\, x: \ |\mathcal{C}(x)| = \infty) \le \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(|\mathcal{C}(x)| = \infty) = \sum_{x\in\mathbb{Z}^d} \theta(p) = 0.$$

(b) If $\theta(p) > 0$, in order to deduce that $\mathbb{P}_p(A_\infty) = 1$, we will use an ergodicity argument.

Recall Kolmogorov's 0-1 law: let (X_i) be i.i.d. random variables, $\mathcal{F}_n = \sigma(X_k : k \ge n)$ and $\mathcal{F}_\infty = \cap_{n\ge 0}\mathcal{F}_n$. Then $\mathcal{F}_\infty$ is trivial, i.e. every $A \in \mathcal{F}_\infty$ satisfies $\mathbb{P}(A) \in \{0,1\}$.

Order the edges of $\mathbb{Z}^d$ and let $\omega(e_1), \omega(e_2), \ldots$ be the states of the ordered edges. Then they are i.i.d. and $A_\infty \in \mathcal{F}_\infty$, since changing the states of finitely many edges does not change the occurrence of A_∞. So $\mathbb{P}_p(A_\infty) \in \{0,1\}$. Since $\theta(p) > 0$ and $\mathbb{P}_p(A_\infty) \ge \theta(p) > 0$, this means that $\mathbb{P}_p(A_\infty) = 1$. $\square$

Once we have established that $\mathbb{P}_p(A_\infty)$ satisfies a $0-1$ law, we ask the following: in the case where $\mathbb{P}_p(A_\infty) = 1$, how many infinite clusters are there?

Theorem 1.14. *Let N be the number of infinite clusters. For all $p > p_c$ we have that*

$$\mathbb{P}_p(N = 1) = 1.$$

Remark 1.15. To prove the above theorem, first we would like to apply an ergodicity argument as in the previous proof. Notice that here we cannot use Kolmogorov's $0-1$ law. Indeed, the event $\{N = k\}$ is not a tail event, since changing just one edge could change its occurrence. Only for $k = 0$ or $k = \infty$ this is the case. However, N is a translation invariant random variable, and hence by Exercise 4 it is almost surely a constant.

Theorem 1.14 was first proved by Aizenman, Kesten and Newman. Here we present the argument by Burton and Keane and we follow the exposition of Grimmett [3] and Nolin [7].

Proof of Theorem 1.14. First by Exercise 4 we have that N is almost surely a constant. Thus there exists $k \in \{0,1,\ldots\} \cup \{\infty\}$ such that $\mathbb{P}_p(N = k) = 1$. We know from Lemma 1.13 that $k \ge 1$. Therefore we will exclude the case where $k \ge 2$ and $k = \infty$.

We first prove that if we assume that $k < \infty$, then $k = 1$.

Let $B(n) = [-n,n]^d \cap \mathbb{Z}^d$ be the d-dimensional box of side length $2n+1$ and let $\partial B(n)$ denote its boundary. Then as $n \to \infty$ we get

$$\mathbb{P}_p(N = k, \text{ all } k \ \infty\text{-clusters intersect } \partial B(n)) \to 1.$$

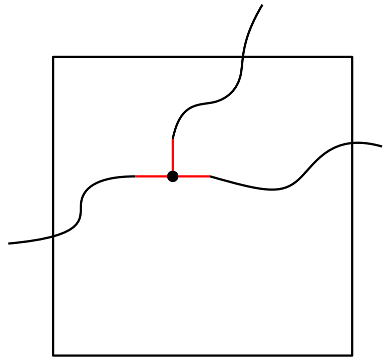


Figure 1: Example of a trifurcation

So there exists n sufficiently large such that

$$\mathbb{P}_p(\text{all } \infty\text{-clusters intersect } \partial B(n)) \geq \frac{1}{2}.$$

Now we note that the event above is independent of the states of the edges in the box $B(n)$. So we obtain

$$\begin{aligned}\mathbb{P}_p(N=1) &\geq \mathbb{P}_p(\text{all } \infty\text{-clusters intersect } \partial B(n),\ \text{all edges of } B(n) \text{ are open}) \\ &= \mathbb{P}_p(\text{all } \infty\text{-clusters intersect } \partial B(n))\,\mathbb{P}_p(\text{all edges of } B(n) \text{ are open}) \geq \frac{1}{2}\cdot p^{E(B(n))},\end{aligned}$$

where $E(B(n))$ is the number of edges in the box $B(n)$. We thus showed that with positive probability $N=1$, but since N is almost surely a constant, this implies that under the assumption that $k<\infty$ we get that $k=1$.

To finish the proof we now have to exclude the possibility $k=\infty$. Suppose that $\mathbb{P}_p(N=\infty)=1$. Let $S(n)$ be the diamond

$$S(n) = \{x \in \mathbb{Z}^d : \|x\|_1 \leq n\},$$

where $\|x\|_1 = \sum_{i=1}^d |x_i|$ for all $x=(x_1,\ldots,x_d)$. The assumption that $N=\infty$ almost surely now gives that as $n\to\infty$

$$\mathbb{P}_p\Big(\text{at least three } \infty\text{-clusters in } \mathbb{Z}^d \setminus S(n) \text{ intersect } \partial S(n)\Big) \to 1.$$

We note again that the event above is independent of the states of the edges in $S(n)$. We now take n sufficiently large so that

$$\mathbb{P}_p\Big(\exists\, \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z}^d \setminus S(n) \text{ intersecting } \partial S(n)\Big) \geq \frac{1}{2}. \tag{1.1}$$

We next introduce the notion of a trifurcation. A vertex x is called a trifurcation if the following three conditions are satisfied:

(i) x belongs to an ∞ open cluster $\mathcal{C}_\infty$,

(ii) there exist exactly three open edges adjacent to x and

(iii) the set $\mathcal{C}_\infty \setminus \{x\}$ contains exactly three infinite clusters and no finite ones.

By translation invariance it is clear that $\mathbb{P}_p(x \text{ is a trifurcation}) = \mathbb{P}_p(0 \text{ is a trifurcation})$ for all x. Our next goal now is to prove that

$$\mathbb{P}_p(0 \text{ is a trifurcation}) \geq c > 0,$$

for some constant c. Let $x_1, x_2, x_3 \in \partial S(n)$ be three distinct points. Then there exist three disjoint paths $(\gamma_i)_{i\leq 3}$ in $S(n)$ joining each x_i to 0. Check! (*This is why we choose to work with $S(n)$ instead of $B(n)$.*) Let $J(x_1, x_2, x_3)$ be the event that all the edges on these three paths are open and all other edges of $S(n)$ are closed.

We can now lower bound the probability that 0 is a trifurcation. To this end we have

$$\mathbb{P}_p(0 \text{ is a trifurcation}) \geq \mathbb{P}_p\Big(\exists\, \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z}^d \setminus S(n) \text{ intersecting } \partial S(n), J(x_1,x_2,x_3)\Big),$$

where $\{x_i\} = \mathcal{C}_\infty^i \cap \partial S(n)$ for $i \leq 3$. Since the points x_i can be chosen in a deterministic way from the configuration outside $S(n)$ and using the fact that the two events above are independent, we obtain

$$\begin{aligned}
&\mathbb{P}_p\Big(\exists\, \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z}^d \setminus S(n) \text{ intersecting } \partial S(n), J(x_1,x_2,x_3)\Big) \\
&= \mathbb{P}_p\Big(J(x_1,x_2,x_3) \;\Big|\; \exists\, \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z}^d \setminus S(n) \text{ intersecting } \partial S(n)\Big) \\
&\qquad\qquad \times \mathbb{P}_p\Big(\exists\, \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z}^d \setminus S(n) \text{ intersecting } \partial S(n)\Big) \geq \min(p,(1-p))^{E(S(n))} \cdot \frac{1}{2},
\end{aligned}$$

where we used (1.1) and again $E(S(n))$ stands for the number of edges in $S(n)$, which is finite. This proves that

$$\mathbb{P}_p(0 \text{ is a trifurcation}) \geq c > 0.$$

We will now arrive at a contradiction. First of all using the above we get

$$\mathbb{E}_p\left[\sum_{x\in B(n)} \mathbf{1}(x \text{ is a trifurcation})\right] \geq c \cdot |B(n)| = c \cdot (2n+1)^d. \tag{1.2}$$

We now claim that almost surely

$$\sum_{x\in B(n)} \mathbf{1}(x \text{ is a trifurcation}) \leq |\partial B(n)|. \tag{1.3}$$

Combining this with (1.2) gives the contradiction, since $|\partial B(n)|$ grows as n^{d-1}. So it only remains to prove (1.3).

Let x_1 be a trifurcation. Then by definition there exist three disjoint self avoiding paths from x_1 to $\partial B(n)$. Let x_2 be another trifurcation. Then there will exist again three disjoint self avoiding paths to $\partial B(n)$. These paths could intersect the paths of x_1, but they cannot create a cycle, since this would violate the definition of a trifurcation. Continuing in this way, we explore all trifurcations, and this gives rise to a forest. The trifurcations are vertices of degree 3 in the forest and the boundary vertices are at the intersection with the boundary of $B(n)$, see Figure 1.3.2. It

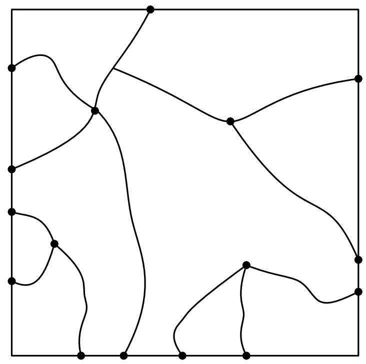


Figure 2: Forest of trifurcations in the box and leaves on the boundary

is easy to show that for any finite tree we have

$$\#\{\text{vertices of degree } 3\} \le \#\,\text{leaves}.$$

Therefore, this shows that the number of trifurcations is upper bounded by the number of leaves, which is clearly upper bounded by the size of $\partial B(n)$ and this concludes the proof. □

1.4 Correlation inequalities

The goal of this section is to state and prove some correlation inequalities for product measure (they are actually more general). The tools that we will develop will be used a lot in the proofs of important results in percolation theory.

Same setup as before, let $G = (V, E)$ be a graph and $\Omega = \{0,1\}^E$ be endowed with the σ-algebra $\mathcal{F}$ generated by the cylinder sets and the probability measure is the product measure $\mathbb{P}_p(\cdot)$ with $p \in [0,1]$.

Definition 1.16. For two configurations ω and ω' we say that ω' is larger than ω and write $\omega' \ge \omega$ if $\omega'(e) \ge \omega(e)$ for all $e \in E$. This defines a partial order on the set Ω.

A random variable X is called **increasing** if whenever $\omega \le \omega'$, then $X(\omega) \le X(\omega')$. It is called **decreasing** if $-X$ is increasing. An event A is called **increasing** if its indicator is increasing.

Example 1.17. *For instance the event $\{|\mathcal{C}(0)| = \infty\}$ is increasing.*

Using the coupling of all percolation processes we can immediately prove

Theorem 1.18. *If N is an increasing random variables and $p_1 \le p_2$, then*

$$\mathbb{E}_{p_1}[N] \le \mathbb{E}_{p_2}[N]\,.$$

Similarly if A is an increasing event, then

$$\mathbb{P}_{p_1}(A) \le \mathbb{P}_{p_2}(A)\,.$$

The first inequality that we will prove says that two increasing events are positively correlated. In other words, conditioning on an increasing event, increases the probability of the occurrence of another increasing event. For instance, suppose we condition on $\{x \leftrightarrow y\}$. Then this increases the probability of the event $\{u \leftrightarrow v\}$.

Theorem 1.19 (FKG inequality)**.** *Let X and Y be two increasing variables on $(\Omega, \mathcal{F})$ such that $\mathbb{E}_p\left[X^2\right] < \infty$ and $\mathbb{E}_p\left[Y^2\right] < \infty$. Then*

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\,\mathbb{E}_p[Y]\,.$$

In particular, if A and B are two increasing events, then

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\,\mathbb{P}_p(B)\,.$$

Remark 1.20. FKG stands for Fortuin, Kasteleyn and Ginibre. Another way of writing the above inequality is that for A and B increasing events one has

$$\mathbb{P}_p(A \mid B) \geq \mathbb{P}_p(A)\,,$$

i.e. knowledge of B increases the probability of A.

Example 1.21. Let G be a graph and for every vertex x we define

$$p_c(x) = \sup\{p \in [0,1] : \mathbb{P}_p(|\mathcal{C}(x)| = \infty) = 0\}.$$

Then as a consequence of the FKG inequality we get that $p_c(x) = p_c(y)$ for all x, y. So this is saying that for bond (same argument for site) percolation the choice of x is irrelevant in the definition of the critical probability.

Proof of the FKG inequality. We start by proving the FKG inequality for random variables X and Y that only depend on a finite set of edges $e_1, \ldots, e_n$. Then we extend to the general case.

We will prove the first claim by induction on n. First suppose that X and Y depend only on the state of the edge e_1, which we call $\omega(e_1)$. Using the increasing property of X and Y we obtain that for all $\omega_1, \omega_2 \in \{0,1\}$

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0.$$

So this immediately gives now

$$\begin{aligned} 0 \leq \sum_{\omega_1,\omega_2} (X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2))\mathbb{P}_p(\omega(e_1) = \omega_1)\,\mathbb{P}_p(\omega(e_1) = \omega_2) \\ = 2(\mathbb{E}_p[XY] - \mathbb{E}_p[X]\,\mathbb{E}_p[Y]). \end{aligned}$$

Suppose now that the claim holds for all $n < k$. Let X and Y be increasing functions of $\omega(e_1), \ldots, \omega(e_k)$. Using properties of conditional expectation we obtain

$$\mathbb{E}_p[XY] = \mathbb{E}_p[\mathbb{E}_p[XY \mid \omega(e_1), \ldots, \omega(e_{k-1})]]\,.$$

We now note that after conditioning on $\omega(e_1), \ldots, \omega(e_{k-1})$ the variables X and Y become increasing functions of $\omega(e_k)$ only, and hence

$$\mathbb{E}_p[XY \mid \omega(e_1), \ldots, \omega(e_{k-1})] \geq \mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})] \, \mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]$$

Applying the induction step to $\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]$ and $\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]$ we get

$$\mathbb{E}_p[\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})] \, \mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]] \geq \mathbb{E}_p[X] \, \mathbb{E}_p[Y] \, .$$

Next we prove it for X and Y that depend on countably many edges and have finite second moments. Let $e_1, \ldots$ be an ordering of the edges of G and for each n define

$$X_n = \mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_n)] \quad \text{and} \quad Y_n = \mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_n)] \, .$$

So now X_n and Y_n are increasing functions of the states of the edges $e_1, \ldots, e_n$, and hence we can apply what we proved earlier to get

$$\mathbb{E}_p[X_n Y_n] \geq \mathbb{E}_p[X_n] \, \mathbb{E}_p[Y_n] \, . \tag{1.4}$$

Since (X_n) and (Y_n) are both martingales bounded in $\mathcal{L}^2$, by the $\mathcal{L}^2$ martingale convergence theorem we deduce as $n \to \infty$

$$X_n \xrightarrow{\mathcal{L}^2} X \quad \text{and} \quad Y_n \xrightarrow{\mathcal{L}^2} Y.$$

In particular, we have convergence in $\mathcal{L}^1$, and hence taking limits in (1.4) we conclude

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X] \, \mathbb{E}_p[Y] \, .$$

This completes the proof. $\square$

Remark 1.22. The FKG inequality gives us a lower bound on the probability of the intersection of two increasing events A and B. Often one also needs an upper bound on this probability. However, such a bound cannot always be useful for any increasing events A and B. It turns out that we can obtain a useful upper bound if instead we look at the event $A \circ B$, which means that A and B occur disjointly, i.e. the set of edges that determines whether or not A occurs is disjoint from the one corresponding to B.

Before giving the formal definition of disjoint occurrence let us start with an example.

Example 1.23. Perform bond percolation on $\mathbb{L}^d$ and let x, y, w, z be 4 distinct points in $\mathbb{Z}^d$. Then we define $\{x \leftrightarrow y\} \circ \{w \leftrightarrow z\}$ to be the event that there exist two disjoint paths one joining x to y and the other one joining w to z. Then if we condition on $\{w \leftrightarrow z\}$, it is intuitive that this should decrease the probability of $\{x \leftrightarrow y\}$. This is exactly what the BK inequality says.

Definition 1.24. For every $\omega \in \Omega = \{0,1\}^E$ and a subset $S \subseteq E$ we write

$$[\omega]_S = \{\omega' \in \Omega : \omega'(e) = \omega(e), \quad \forall \, e \in S\}.$$

Let A and B be two events depending on a finite set of edges F. Then the disjoint occurrence of A

and B is defined via

$$A \circ B = \big\{ \omega \in \Omega : \exists S \subseteq F \text{ with } [\omega]_S \subseteq A \text{ and } [\omega]_{F \setminus S} \subseteq B \big\} .$$

Theorem 1.25 (BK inequality)**.** *Let F be a finite set and $\Omega = \{0,1\}^F$. For all increasing events A and B on Ω we have*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\, \mathbb{P}_p(B) .$$

Proof. This theorem was proved by van den Berg and Kesten (see [3]). Here we present an easier proof given by Bollobás and Leader (see for instance [1]).

The proof follows by induction on the size n of the set F. For $n = 0$, this trivially holds. Suppose that it works for all sets of size $n-1$, we will show it also works for size n. For every set $D \subseteq \{0,1\}^F$ we define for $i = 0, 1$

$$D_i = \{(\omega_1, \ldots, \omega_{n-1}) : (\omega_1, \ldots, \omega_{n-1}, i) \in D\}.$$

By the independence of different coordinates we obtain

$$\mathbb{P}_p(D) = p\mathbb{P}_p(D_1) + (1-p)\mathbb{P}_p(D_0) . \tag{1.5}$$

Let $A, B \subseteq \{0,1\}^F$ and $C = A \circ B$. Then it is easy to check that

$$C_0 = A_0 \circ B_0 \text{ and } C_1 = (A_1 \circ B_0) \cup (A_0 \circ B_1).$$

Also since A is increasing, we get $A_0 \subseteq A_1$ and both A_0 and A_1 are increasing events. Similarly for B. Therefore,

$$C_0 \subseteq (A_0 \circ B_1) \cap (A_1 \circ B_0) \text{ and } C_1 \subseteq A_1 \circ B_1.$$

Using this and the induction hypothesis we get that

$$\begin{aligned}
\mathbb{P}_p(C_0) &= \mathbb{P}_p(A_0 \circ B_0) \leq \mathbb{P}_p(A_0)\, \mathbb{P}_p(B_0) \\
\mathbb{P}_p(C_1) &\leq \mathbb{P}_p(A_1 \circ B_1) \leq \mathbb{P}_p(A_1)\, \mathbb{P}_p(B_1) \\
\mathbb{P}_p(C_0) + \mathbb{P}_p(C_1) &\leq \mathbb{P}_p((A_0 \circ B_1) \cap (A_1 \circ B_0)) + \mathbb{P}_p((A_1 \circ B_0) \cup (A_0 \circ B_1)) \\
&= \mathbb{P}_p(A_0 \circ B_1) + \mathbb{P}_p(A_1 \circ B_0) \leq \mathbb{P}_p(A_0)\, \mathbb{P}_p(B_1) + \mathbb{P}_p(A_1)\, \mathbb{P}_p(B_0) .
\end{aligned}$$

Multiplying the first inequality by $(1-p)^2$, the second p^2 and the third one by $p(1-p)$. Adding them all up we obtain

$$(1-p)\mathbb{P}_p(C_0) + p\mathbb{P}_p(C_1) \leq (p\mathbb{P}_p(A_1) + (1-p)\mathbb{P}_p(A_0))\, (p\mathbb{P}_p(B_1) + (1-p)\mathbb{P}_p(B_0)) .$$

From (1.5) we finally get

$$\mathbb{P}_p(C) \leq \mathbb{P}_p(A)\, \mathbb{P}_p(B)$$

and this concludes the proof. $\square$

After the BK inequality was proved, it was conjectured that it should hold more generally, not only for increasing events A and B. This was later proved by Reimer. Despite being more general (it actually includes both FKG and BK inequality) it has not found many applications, i.e. instances where the other inequalities cannot be used.

Theorem 1.26 (Reimer's inequality)**.** *Let F be a finite set and $\Omega = \{0,1\}^F$. For all A and B we have*

$$\mathbb{P}_p(A \circ B) \le \mathbb{P}_p(A)\,\mathbb{P}_p(B)\,.$$

We end this section by presenting an application of the BK inequality. Let $\chi(p) = \mathbb{E}_p[|\mathcal{C}(0)|]$ be the mean cluster size. Recall $\mathcal{B}_n = [-n,n]^d \cap \mathbb{Z}^d$.

Theorem 1.27. *Suppose that $\chi(p) < \infty$. Then there exists a positive constant c so that for all $n \ge 1$ we have*

$$\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_n) \le e^{-cn}.$$

This theorem was proved by Hammersley using a different method, but the use of the BK inequality makes the proof much shorter. In a later section we will prove the same statement under the weaker assumption that $p < p_c$. The reason we chose to present the proof of the above result is because it is a very nice application of the BK inequality and also to illustrate a method which is commonly used.

Proof of Theorem 1.27. Let $X_n = \sum_{x \in \mathcal{B}_n} \mathbf{1}(0 \leftrightarrow x)$ be the number of vertices on $\partial \mathcal{B}_n$ that are connected to 0 via an open path of edges. Then

$$\sum_{n=0}^{\infty} \mathbb{E}_p[X_n] = \sum_{n=0}^{\infty} \sum_{x \in \partial \mathcal{B}_n} \mathbb{P}_p(0 \leftrightarrow x) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow x) = \chi(p).$$

If there exists an open path of edges from 0 to $\partial \mathcal{B}_{m+k}$, then there must exist a vertex x on $\partial \mathcal{B}_m$ which is connected to 0 and $\partial \mathcal{B}_{m+k}$ with two disjoint paths. Therefore, using the BK inequality this gives for all $m, k \ge 1$

$$\begin{aligned}
\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_{m+k}) &\le \sum_{x \in \partial \mathcal{B}_m} \mathbb{P}_p(0 \leftrightarrow x)\,\mathbb{P}_p(x \leftrightarrow \partial \mathcal{B}_{m+k}) \\
&\le \sum_{x \in \partial \mathcal{B}_m} \mathbb{P}_p(0 \leftrightarrow x)\,\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_k) = \mathbb{E}_p[X_m]\,\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_k)\,,
\end{aligned}$$

where in the last step we used translation invariance. Since we assumed that $\chi(p) < \infty$, this means that $\mathbb{E}_p[X_n] \to 0$ as $n \to \infty$, and hence there exists m large so that

$$\mathbb{E}_p[X_m] < \delta < 1.$$

Now for any $n \in \mathbb{N}$ with $n \ge m$ we can find $q, r \ge 0$ with $r \in [0, m-1]$ such that $n = qm + r$. Then we finally obtain

$$\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_n) \le \mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_{qm}) \le \delta^q \le \delta^{-1+n/m} = e^{-cn}$$

for a suitable constant $c > 0$. By adjusting the value of c if necessary to take care of values of $n < m$, this concludes the proof. $\square$

1.5 Russo's formula

We saw in Lemma 1.2 that using the coupling of all percolation processes, the probability $\theta(p)$ is an increasing function of p. But, do we know the rate at which it changes when we increase p? This is the goal of the current section, to understand the derivative of $\mathbb{P}_p(A)$ for all increasing events A as a function of p.

Before stating Russo's theorem we define what we mean by a pivotal edge.

Definition 1.28. Let A be an event and ω a percolation configuration. An edge e is called *pivotal* for (A, ω) if $\mathbf{1}(\omega \in A) \neq \mathbf{1}(\omega' \in A)$, where $\omega'(f) = \omega(f)$ for all edges $f \neq e$ and $\omega'(e) \neq \omega(e)$. In other words, e is pivotal for A if the occurrence of A depends crucially on whether the edge e is present or not.

The event $\{e \text{ is pivotal for } A\}$ is equal to $\{\omega : e \text{ is pivotal for } (A, \omega)\}$.

Example 1.29. Let A be the event that 0 is in an infinite cluster. Then an edge e is pivotal for A if the removal of e leads to a finite component containing the origin and one endpoint of e and an infinite component containing the other endpoint of e.

Theorem 1.30 (Russo's formula)**.** *Let A be an increasing event that depends only on the states of a finite number of edges. Then*

$$\frac{d}{dp}\mathbb{P}_p(A) = \mathbb{E}_p[N(A)]\,,$$

where $N(A)$ is the number of pivotal edges for A.

Proof. First of all we argue that the function in question is indeed differentiable. Suppose that A depends on the states of m edges, $e_1, \ldots, e_m$. We define the function

$$f(p_1, \ldots, p_m) = \mathbb{P}_{\overline{p}}(A)\,,$$

where $\overline{p} = (p_1, \ldots, p_m)$ and p_i is the probability that the i-th edge is open. Then clearly f is a polynomial in p_i's, and hence it is differentiable.

We now turn to establish the formula for the derivative. Recall the coupling of all percolation processes. Let $(X(e))_{e \in \mathbb{L}^d}$ be i.i.d. uniform random variables in $[0,1]$. For a vector of probabilities $\overline{p} = (p(e) : e \in \mathbb{L}^d)$ we write $\eta_{\overline{p}}(e) = 1$ if $X(e) \leq p(e)$ and $\eta_{\overline{p}}(e) = 0$ otherwise. We write $\mathbb{P}_{\overline{p}}(\cdot)$ for the probability measure where edge e is open with probability $p(e)$ and closed with $1 - p(e)$ independently over different edges. So we have

$$\mathbb{P}_{\overline{p}}(A) = \mathbb{P}(\eta_{\overline{p}} \in A)\,.$$

Fix an edge f and let $\overline{p}'$ be such that $p'(e) = p(e)$ for all $e \neq f$ and $p'(f) = p(f) + \delta$ with $\delta > 0$. Then we have

$$\mathbb{P}_{\overline{p}'}(A) - \mathbb{P}_{\overline{p}}(A) = \mathbb{P}\big(\eta_{\overline{p}'} \in A\big) - \mathbb{P}(\eta_{\overline{p}} \in A) = \mathbb{P}\big(\eta_{\overline{p}} \notin A,\, \eta_{\overline{p}'} \in A\big)\,,$$

where the last equality follows from the increasing property of A. We now claim that

$$\mathbb{P}\big(\eta_{\overline{p}} \notin A,\, \eta_{\overline{p}'} \in A\big) = \delta \cdot \mathbb{P}_{\overline{p}}(f \text{ is pivotal for } A)\,. \tag{1.6}$$

Since $\overline{p}'$ agrees with $\overline{p}$ everywhere except for the edge f, on the event $\{\eta_{\overline{p}} \notin A,\, \eta_{\overline{p}'} \in A\}$ we have $\eta_{\overline{p}}(f) = 0$ while $\eta_{\overline{p}'}(f) = 1$. Hence this means that f is pivotal for the event A. We also notice that the event that f is pivotal for A is independent of the state of the edge f. Therefore we get

$$\begin{aligned}\mathbb{P}\big(\eta_{\overline{p}} \notin A,\, \eta_{\overline{p}'} \in A\big) &= \mathbb{P}_{\overline{p}}(f \text{ is pivotal for } A,\, p < X(f) \leq p+\delta) \\ &= \mathbb{P}(p < X(f) \leq p+\delta)\, \mathbb{P}_{\overline{p}}(f \text{ is pivotal for } A) = \delta \cdot \mathbb{P}_{\overline{p}}(f \text{ is pivotal for } A)\,,\end{aligned}$$

which is exactly (1.6). So we can now conclude

$$\frac{\partial}{\partial p(f)} \mathbb{P}_{\overline{p}}(A) = \lim_{\delta \downarrow 0} \frac{\mathbb{P}_{\overline{p}'}(A) - \mathbb{P}_{\overline{p}}(A)}{\delta} = \mathbb{P}_{\overline{p}}(f \text{ is pivotal for } A)\,.$$

Up to here we have not used the assumption that A depends only on finitely many edges. We only used the fact that A is increasing. Since A depends on finitely many edges, say $e_1, \ldots, e_m$, then as we said above $\mathbb{P}_p(A)$ is a polynomial of $p(e_1), \ldots, p(e_m)$, and hence by the chain rule we obtain

$$\frac{d}{dp} \mathbb{P}_p(A) = \sum_{i=1}^{m} \frac{\partial}{\partial p(e_i)} \mathbb{P}_{\overline{p}}(A)\,|_{\overline{p}=(p,\ldots,p)} = \sum_{i=1}^{m} \mathbb{P}_p(e_i \text{ is pivotal for } A) = \mathbb{E}_p[N(A)]$$

and this completes the proof. $\square$

Remark 1.31. We note that if A is an increasing event depending on an infinite number of edges, then

$$\liminf_{\delta \downarrow 0} \frac{\mathbb{P}_{p+\delta}(A) - \mathbb{P}_p(A)}{\delta} \geq \mathbb{E}_p[N(A)]\,.$$

Indeed, let $\mathcal{B}_n = [-n, n]^d \cap \mathbb{Z}^d$ and define $\overline{p}_n$ via

$$\overline{p}_n(e) = \begin{cases} p & \text{if } e \notin \mathcal{B}_n \\ p+\delta & \text{if } e \in \mathcal{B}_n. \end{cases}$$

Since A is an increasing event we now get

$$\frac{\mathbb{P}_{p+\delta}(A) - \mathbb{P}_p(A)}{\delta} \geq \frac{\mathbb{P}_{\overline{p}_n}(A) - \mathbb{P}_p(A)}{\delta}.$$

If $\overline{q}$ and $\overline{p}$ only differ at the value of only one edge e by δ and $\overline{q} \leq \overline{p}$, then by (1.6) we obtain

$$\mathbb{P}_{\overline{p}}(A) - \mathbb{P}_{\overline{q}}(A) = \delta \cdot \mathbb{P}_{\overline{q}}(e \text{ is pivotal for } A)\,. \tag{1.7}$$

Also for any event B if $\overline{p}$ and $\overline{q}$ differ in a finite number m of edges, then

$$\begin{aligned}|\mathbb{P}_{\overline{q}}(B) - \mathbb{P}_{\overline{p}}(B)\,| &= |\mathbb{P}(\eta_{\overline{q}} \in B) - \mathbb{P}(\eta_{\overline{p}} \in B)| \\ &\leq \mathbb{P}(\exists \text{ edge } e :\, \eta_{\overline{p}}(e) \neq \eta_{\overline{q}}(e)) \leq m \cdot \mathbb{P}(X(e) \in [p(e) \wedge q(e), p(e) \vee q(e)])\,.\end{aligned} \tag{1.8}$$

So by taking successive differences and using (1.7) and (1.8) this gives

$$\liminf_{\delta \downarrow 0} \frac{\mathbb{P}_{\overline{p}_n}(A) - \mathbb{P}_p(A)}{\delta} \geq \sum_{e \in \mathcal{B}_n} \mathbb{P}_p(e \text{ is pivotal for } A)\,.$$

Taking the limit as $n \to \infty$ completes the proof.

Corollary 1.32. *Let A be an increasing event depending on the states of m edges and $p \le q$ be in $(0,1]$. Then*

$$\mathbb{P}_q(A) \le \left(\frac{q}{p}\right)^m \mathbb{P}_p(A)\,.$$

Proof. As already mentioned in the proof of Russo's formula, the event that the edge e is a pivotal edge is independent of its state, i.e.

$$\mathbb{P}_p(e \text{ is open and pivotal for } A) = p \cdot \mathbb{P}_p(e \text{ is pivotal for } A)\,.$$

Moreover, since A is increasing, we have that

$$\{\omega : \omega(e) = 1 \text{ and } \mathbf{1}(\omega \in A) \neq \mathbf{1}(\omega' \in A)\} = \{\omega : e \text{ is pivotal for } (A, \omega) \text{ and } A \text{ occurs}\},$$

where ω' agrees with ω everywhere except for the edge e. Using these two equalities above we thus get from Russo's formula

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_p(A) &= \frac{1}{p} \cdot \sum_e \mathbb{P}_p(e \text{ is open and pivotal for } A) \\
&= \frac{1}{p} \cdot \sum_e \mathbb{P}_p(e \text{ is pivotal for } A,\, A) = \frac{1}{p} \cdot \sum_e \mathbb{P}_p(e \text{ e is pivotal for } A \mid A)\, \mathbb{P}_p(A) \\
&= \frac{1}{p} \cdot \mathbb{E}_p[N(A) \mid A]\, \mathbb{P}_p(A)\,.
\end{aligned}$$

Dividing through by $\mathbb{P}_p(A)$ and integrating from p to q we obtain

$$\mathbb{P}_q(A) = \mathbb{P}_p(A) \exp\left(\int_p^q \frac{1}{s} \cdot \mathbb{E}_s[N(A) \mid A]\; ds\right).$$

Using the trivial bound

$$\mathbb{E}_p[N(A) \mid A] = \sum_e \mathbb{P}_p(e \text{ is pivotal for } A) \le m$$

proves the corollary. □

1.6 Subcritical phase

In this section we focus on $p < p_c$. In this case we know that there is no infinite cluster almost surely. However, one can ask what is the size of the cluster of 0. How do probabilities like $\mathbb{P}_p(|\mathcal{C}(0)| \ge n)$ decay in n?

We write $\mathcal{B}_n = [-n, n]^d \cap \mathbb{Z}^d$.

Theorem 1.33. *Let $d \ge 2$. Then the following are true*

(a) *If $p < p_c$, then there exists a constant c so that for all $n \ge 1$ we have*

$$\mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n) \le e^{-cn}.$$

(b) *If* $p > p_c$, *then*

$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) \geq \frac{p - p_c}{p(1-p_c)}.$$

This theorem was first proved by Aizenman and Barsky in a more general framework of long range percolation. A different proof was discovered by Menshikov at the same time using the geometry of the pivotal edges. Here we will present a new proof of Duminil-Copin and Tassion from last year, which is much shorter than the previous ones. We follow closely [2].

Proof. The main idea of the proof is to introduce a different critical probability that is more amenable to calculations. First, for all finite sets S containing 0 we define

$$\varphi_p(S) = p \sum_{(x,y)\in\partial S} \mathbb{P}_p\Big(0 \stackrel{S}{\longleftrightarrow} x\Big),$$

where we write $x \stackrel{S}{\longleftrightarrow} y$ if x is connected to y via an open path of edges all lying in the set S and $\partial S = \{(x,y) \in E : x \in S \text{ and } y \notin S\}$. We now define

$$\widetilde{p_c} = \sup\{p \in [0,1] : \exists\, \text{a finite set } S \ni 0 \,\text{with}\, \varphi_p(S) < 1\}.$$

Since for any finite set S the function $\varphi_p(S)$ is continuous in p, it follows that the set of p's for which there exists a finite set $S \ni 0$ with $\varphi_p(S) < 1$ is an open subset of $[0,1]$. Therefore, $\widetilde{p_c} > 0$. We now notice that it suffices to prove the statement of the theorem with p_c replaced by $\widetilde{p_c}$. Indeed, from part (a) we would get that $\widetilde{p_c} \leq p_c$, since for all $p < \widetilde{p_c}$ by letting $n \to \infty$ would give $\theta(p) = 0$. Therefore, $\widetilde{p_c} < 1$, and hence from part (b) we would get that $p_c \leq \widetilde{p_c}$.

So we now turn to prove (a) and (b) with p_c replaced by $\widetilde{p_c}$.

(a) Let $p < \widetilde{p_c}$. Then by the definition there exists a finite set S with $0 \in S$ and such that $\varphi_p(S) < 1$. Take L large so that $S \subseteq \mathcal{B}_{L-1}$. Let $k \geq 1$ and assume that the event $\{0 \longleftrightarrow \partial\mathcal{B}_{kL}\}$ holds. Let

$$\mathcal{C} = \{x \in S : 0 \stackrel{S}{\longleftrightarrow} x\}.$$

Since $S \cap \partial\mathcal{B}_{Lk} = \varnothing$, on the event $\{0 \longleftrightarrow \partial\mathcal{B}_{kL}\}$ there must exist an edge $(x,y) \in \partial S$ such that $0 \stackrel{S}{\longleftrightarrow} x$, the edge (x,y) is open and $y \stackrel{\mathcal{C}^c}{\longleftrightarrow} \partial\mathcal{B}_{kL}$. So we obtain

$$\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_{kL}) \leq \sum_{(x,y)\in\partial S} \sum_{A\subseteq S} \mathbb{P}_p\Big(0 \stackrel{S}{\longleftrightarrow} x, (x,y) \text{ is open}, \mathcal{C} = A, y \stackrel{A^c}{\longleftrightarrow} \partial\mathcal{B}_{kL}\Big).$$

We now notice that the events $\{0 \stackrel{S}{\longleftrightarrow} x, \mathcal{C} = A\}$, $\{(x,y) \text{ is open}\}$ and $\{y \stackrel{A^c}{\longleftrightarrow} \partial\mathcal{B}_{kL}\}$ appearing above are independent, since they depend on disjoint sets of edges. (This is a trivial application of the BK inequality.) Therefore we deduce

$$\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_{kL}) \leq p \sum_{(x,y)\in\partial S} \sum_{A\subseteq S} \mathbb{P}_p\Big(0 \stackrel{S}{\longleftrightarrow} x, \mathcal{C} = A\Big)\, \mathbb{P}_p\Big(y \stackrel{A^c}{\longleftrightarrow} \partial\mathcal{B}_{kL}\Big).$$

Since $y \in \mathcal{B}_L$, it follows that

$$\mathbb{P}_p\Big(y \stackrel{A^c}{\longleftrightarrow} \partial\mathcal{B}_{kL}\Big) \leq \mathbb{P}_p\big(0 \longleftrightarrow \partial\mathcal{B}_{(k-1)L}\big),$$

and hence we conclude

$$\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_{kL}) \le p \sum_{(x,y)\in\partial S} \mathbb{P}_p\Big(0 \overset{S}{\longleftrightarrow} x\Big)\, \mathbb{P}_p\big(0 \longleftrightarrow \partial\mathcal{B}_{(k-1)L}\big) = \varphi_p(S)\mathbb{P}_p\big(0 \longleftrightarrow \partial\mathcal{B}_{(k-1)L}\big)\,.$$

Iterating the above inequality gives

$$\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_{kL}) \le (\varphi_p(S))^{k-1}\,,$$

and hence this establishes the exponential decay.

(b) To prove the second part it suffices to prove the following differential inequality for all $p \in (0,1)$ and all $n \ge 1$

$$\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_n) \ge \frac{1}{p(1-p)} \cdot \inf_{\substack{S\subseteq\mathcal{B}_n\\ 0\in S}} \varphi_p(S) \cdot (1 - \mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_n)). \tag{1.9}$$

Indeed, once we show this inequality, then integrating from $\widetilde{p_c}$ to $p > \widetilde{p_c}$ and then taking the limit as $n \to \infty$ proves the desired inequality. So we turn to prove (1.9). By Russo's formula since $\{0 \longleftrightarrow \partial\mathcal{B}_n\}$ is increasing and only depends on a finite set of edges, we get

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_n) &= \sum_{e\in\mathcal{B}_n} \mathbb{P}_p(e \text{ is pivotal for } \{0 \longleftrightarrow \partial\mathcal{B}_n\}) \\ &= \frac{1}{1-p}\sum_{e\in\mathcal{B}_n} \mathbb{P}_p(e \text{ is pivotal, } 0 \not\longleftrightarrow \partial\mathcal{B}_n)\,,\end{aligned}$$

since the events $\{e \text{ is pivotal}\}$ and $\{e \text{ is closed}\}$ are independent and also

$$\{e \text{ is pivotal and closed}\} = \{e \text{ is pivotal, } 0 \not\longleftrightarrow \partial\mathcal{B}_n\}.$$

We next define the set

$$\mathcal{S} = \{x \in \mathcal{B}_n : x \not\longleftrightarrow \partial\mathcal{B}_n\}.$$

On the event $\{0 \not\longleftrightarrow \partial\mathcal{B}_n\}$, the set $\mathcal{S}$ is a subset of $\mathcal{B}_n$ containing 0. So we get

$$\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_n) = \frac{1}{1-p}\sum_{e\in\mathcal{B}_n}\sum_{\substack{A\subseteq\mathcal{B}_n\\ 0\in A}} \mathbb{P}_p(e \text{ is pivotal, } \mathcal{S} = A)\,.$$

We now observe that on the event $\{\mathcal{S} = A\}$, the pivotal edges for the event $\{0 \leftrightarrow \partial\mathcal{B}_n\}$ are the edges $(x,y) \in \partial A$ such that $0 \overset{A}{\longleftrightarrow} x$. Therefore we obtain

$$\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_n) = \frac{1}{1-p}\sum_{\substack{A\subseteq\mathcal{B}_n\\ 0\in A}}\sum_{(x,y)\in\partial A} \mathbb{P}_p\Big(0 \overset{A}{\longleftrightarrow} x, \mathcal{S} = A\Big)\,.$$

Now the two events appearing above are independent, since in order to determine whether $\mathcal{S} = A$, we only need to look at what happens outside of A. Hence this now gives

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\mathcal{B}_n) &= \frac{1}{1-p}\sum_{\substack{A\subseteq\mathcal{B}_n\\0\in A}}\sum_{(x,y)\in\partial A}\mathbb{P}_p\Big(0 \overset{A}{\longleftrightarrow} x\Big)\,\mathbb{P}_p(\mathcal{S}=A)\\ &= \frac{1}{p(1-p)}\sum_{\substack{A\subseteq\mathcal{B}_n\\0\in A}}\varphi_p(A)\mathbb{P}_p(\mathcal{S}=A) \geq \frac{1}{p(1-p)}\inf_{\substack{S\subseteq\mathcal{B}_n\\0\in S}}\varphi_p(S)\,\mathbb{P}_p(0 \not\longleftrightarrow \partial\mathcal{B}_n)\end{aligned}$$

and this concludes the proof. □

Remark 1.34. Note that the assumption of Theorem 1.33 is that $p < p_c$ and not that $\theta(p) = 0$. We will see later that for $d = 2$, at the critical probability $1/2$ we do not have exponential decay of this probability.

Remark 1.35. Note that the probability $\mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n)$ is at least p^n, and hence we cannot hope for a faster convergence than exponential as in Theorem 1.33. As a consequence of Theorem 1.33 we get that when $p < p_c$, then

$$\chi(p) = \mathbb{E}_p[|\mathcal{C}(0)|] < \infty.$$

Theorem 1.33 also gives us an upper bound on the tail probabilities of the distribution of $|\mathcal{C}(0)|$. In particular, we get for $p < p_c$

$$\mathbb{P}_p(|\mathcal{C}(0)| \geq n) \leq \mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_{n^{1/d}}) \leq \exp\left(-cn^{1/d}\right).$$

However, this is not the best possible result, as the term $n^{1/d}$ in the exponential can be replaced by n (see for instance [3, Theorem 6.75]). So if we restrict bond percolation on a large box $\mathcal{B}_n$, i.e. by taking n large enough, then the largest connected cluster will have size of order $\log n$.

1.7 Supercritical phase in $\mathbb{Z}^2$

Conjecture 1.36. *For all $d \geq 2$ we have $\theta(p_c(d)) = 0$.*

This has only been proven for $d = 2$ and $d \geq 11$. We now present the proof for $d = 2$.

Theorem 1.37. *For bond percolation on $\mathbb{Z}^2$ the critical probability $p_c = 1/2$ and $\theta(1/2) = 0$.*

Proof. Different parts of this theorem were proved by Harris, Kesten, Russo, Seymour and Welsh. Here we follow the proof given in Grimmett [3].

We will first prove that $\theta(1/2) = 0$, which will then imply that $p_c \geq 1/2$. Next we will show that $p_c \leq 1/2$. Both parts rely crucially on duality, which is only present in two dimensions.

Suppose that $\theta(1/2) > 0$. Let $S_n = [0,n]^2$. Then there exists n sufficiently large so that

$$\mathbb{P}(\partial S_n \leftrightarrow \infty) > 1 - \left(\frac{1}{8}\right)^4. \tag{1.10}$$

Let A_ℓ, A_t, A_r, A_b be the events that there exists an open path joining the left, respectively top, right and bottom, side of the box S_n to ∞. By symmetry they all have the same probability. Also clearly they are increasing events, and hence by the FKG inequality we obtain

$$\mathbb{P}_{1/2}(\partial S_n \not\leftrightarrow \infty) = \mathbb{P}(A_\ell^c \cap A_t^c \cap A_r^c \cap A_b^c) \geq \left(\mathbb{P}_{1/2}(A_\ell^c)\right)^4 = \left(1 - \mathbb{P}_{1/2}(A_\ell)\right)^4.$$

Therefore, taking n large enough so that (1.10) holds we get that

$$\left(1 - \mathbb{P}_{1/2}(A_\ell)\right)^4 \leq \left(\frac{1}{8}\right)^4,$$

which of course is equivalent to saying

$$\mathbb{P}_{1/2}(A_\ell) \geq \frac{7}{8}.$$

Recall the definition of the dual lattice. Consider the dual box $T_n = [0, n-1]^2 + (1/2, 1/2)$ and let D_ℓ, D_t, D_r, D_b be the events that the left (resp, top, right, bottom) side of T_n is connected to ∞ via a closed path of edges of the dual lattice.

Let $G = A_\ell \cap A_r \cap D_t \cap D_b$. Then by symmetry $\mathbb{P}_{1/2}(D_i) = \mathbb{P}_{1/2}(A_i)$ for all $i \in \{\ell, r, t, b\}$, and hence

$$\mathbb{P}_{1/2}(G^c) \leq \mathbb{P}(A_\ell^c) + \mathbb{P}(A_r^c) + \mathbb{P}(D_t^c) + \mathbb{P}(D_b^c) \leq \frac{1}{2},$$

or equivalently $\mathbb{P}_{1/2}(G) \geq 1/2$. Using the uniqueness of the infinite open cluster above p_c implies that there must exist a connection between the two infinite open clusters. However, this in turn implies that there exist two infinite closed dual paths, since they have to be separated by the open path of the primal. But this has probability 0, by the uniqueness theorem again. We thus reached a contradiction, so $\theta(1/2) = 0$.

We now turn to prove that $p_c \leq 1/2$. Suppose for contradiction that $p_c > 1/2$. Then by the exponential decay in the subcritical regime, Theorem 1.33, there exists a positive constant c such that for all $n \geq 1$

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial \mathcal{B}_n) \leq e^{-cn}. \tag{1.11}$$

Let A_n be the event that the left side of the box $C_n = [0, n+1] \times [0, n]$ is joined to the right side along an open path.

Consider next the dual box $D_n = \{x + (1/2, 1/2) : 0 \leq x_1 \leq n, -1 \leq x_2 \leq n\}$ and let B_n be the event that there exists a closed path of dual edges joining a vertex on the top of the box to the bottom. It is not hard to see that $A_n \cap B_n = \varnothing$ and also that A_n and B_n partition the whole space. Indeed, if both A_n and B_n occur, then there must exist an open edge which is crossed by a closed edge of the dual, which is impossible. Also suppose that A_n does not occur, then there must exist a closed path from top to bottom blocking all connections of the left side to the right side of C_n. Therefore,

$$\mathbb{P}_p(A_n) + \mathbb{P}_p(B_n) = 1$$

and since the dual lattice is isomorphic to $\mathbb{Z}^2$ we also get $\mathbb{P}_p(A_n) = \mathbb{P}_{1-p}(B_n)$. Therefore,

$$\mathbb{P}_{1/2}(A_n) = 1/2. \tag{1.12}$$

However, using the exponential decay below criticality, we obtain for a positive constant c

$$\mathbb{P}_{1/2}(0 \leftrightarrow \{(n,k) : k \in \mathbb{Z}\}) \leq e^{-cn}.$$

So letting $L_n = \{(n,k) : k \in \mathbb{Z}\}$, we deduce

$$\mathbb{P}_{1/2}(A_n) \leq \sum_{k=0}^{n} \mathbb{P}_{1/2}((0,k) \leftrightarrow L_n) \leq (n+1)e^{-cn},$$

but this gives a contradiction to the fact that $\mathbb{P}_{1/2}(A_n) = 1/2$ and thus completes the proof. $\square$

1.8 Russo Seymour Welsh theorem

We saw in the proof of $p_c(2) = 1/2$ the use of open crossings of boxes. We will now see the most fundamental result of RSW which deals with crossings of rectangles.

Let us first fix some notation. First of all we write

$$\mathcal{B}(k\ell, \ell) = [-\ell, (2k-1)\ell] \times [-\ell, \ell]$$

and $\mathcal{B}(\ell, \ell) = \mathcal{B}(\ell) = [-\ell, \ell]^2$. A left-right crossing of a rectangle B is an open path joining the left side of the rectangle to the right side, but it is not allowed to use any of the boundary edges. (This is simply for a technical reason.) We denote by $\mathrm{LR}(k\ell, \ell)$ the event that there exists a left to right crossing of $\mathcal{B}(k\ell, \ell)$ and we write $\mathrm{LR}(\ell)$ for a crossing of $\mathcal{B}(\ell)$. We also denote by

$$A(\ell) = \mathcal{B}(3\ell) \setminus \mathcal{B}(\ell)$$

and we write $O(\ell)$ for the event that there is an open circuit in $A(\ell)$ containing 0 in its interior.

Theorem 1.38 (RSW)**.** *Suppose that $\mathbb{P}_p(\mathrm{LR}(\ell)) = \alpha$. Then*

$$\mathbb{P}_p(O(\ell)) \geq \left(\alpha \left(1 - \sqrt{1-\alpha}\right)^4\right)^{12}.$$

Remark 1.39. Note that the importance of the above theorem is not on the particular lower bound that it gives, but rather that if we have a positive probability of a left to right crossing of the box, then there is a positive probability of having an open circuit in the annulus. We will see some applications of this to inequalities for $\theta(p)$ in the next section.

The proof of RSW is split into two unequal lemmas.

Lemma 1.40. *Suppose that $\mathbb{P}_p(\mathrm{LR}(\ell)) = \alpha$. Then*

$$\mathbb{P}_p\left(\mathrm{LR}\left(\frac{3}{2}\ell, \ell\right)\right) \geq \left(1 - \sqrt{1-\alpha}\right)^3.$$

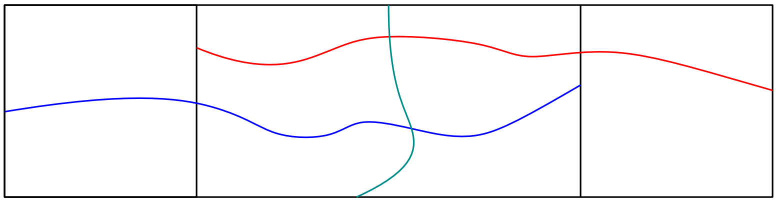


Figure 3: Crossings intersecting

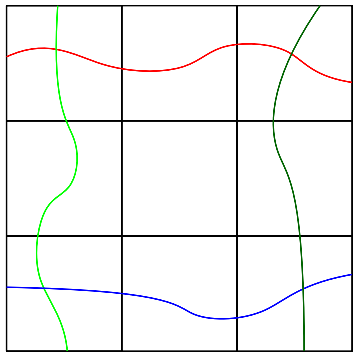


Figure 4: Open circuit in the annulus

Lemma 1.41. *We have that*

$$\mathbb{P}_p(\mathrm{LR}(2\ell,\ell)) \geq \mathbb{P}_p(\mathrm{LR}(\ell)) \left(\mathbb{P}_p\left(\mathrm{LR}\left(\frac{3}{2}\ell,\ell\right)\right)\right)^2 \tag{1.13}$$

$$\mathbb{P}_p(\mathrm{LR}(3\ell,\ell)) \geq \mathbb{P}_p(\mathrm{LR}(\ell)) \left(\mathbb{P}_p(\mathrm{LR}\left(2\ell,\ell\right))\right)^2 \tag{1.14}$$

$$\mathbb{P}_p(O(\ell)) \geq \left(\mathbb{P}_p(\mathrm{LR}(3\ell,\ell))\right)^4 \tag{1.15}$$

Proof of Theorem 1.38. The proof of RSW follows immediately by combining the two lemmas above. □

Proof of Lemma 1.41. This is the argument of Russo. We start by proving (1.13).

Let LR_1 be the event that there exists a left to right crossing of the rectangle $[0,3\ell]\times[-\ell,\ell]$, LR_2 be the event that there exists a left to right crossing of $[\ell,4\ell]\times[-\ell,\ell]$ and TB_1 the event that there exists a top to bottom crossing of $[\ell,3\ell]\times[-\ell,\ell]$. If all these events occur, then it is clear that there exists a left to right crossing of the rectangle $[0,4\ell]\times[-\ell,\ell]$ (see Figure 1.8), and hence we obtain

$$\mathbb{P}_p(\mathrm{LR}(2\ell,\ell)) \geq \mathbb{P}_p(\mathrm{LR}_1\cap\mathrm{LR}_2\cap\mathrm{TB}_1) \geq \mathbb{P}_p(\mathrm{LR}_1)\,\mathbb{P}_p(\mathrm{LR}_2)\,\mathbb{P}_p(\mathrm{TB}_1)\,,$$

where the last inequality follows from the FKG inequality. So this proves (1.13). Inequality (1.14) follows similarly.

For the last inequality we express the event $A(\ell)$ as the union of four rectangles and we ask for left to right crossings of each of those (see Figure 1.8). Then by the FKG inequality again we get the result. □

Sketch of proof of Lemma 1.40. First of all one can define a partial order on left to right crossings of the box $\mathcal{B}(\ell)$. We say $\pi_1 \le \pi_2$ if π_1 is contained in the closed bounded region of $\mathcal{B}(\ell)$ below π_2. It can be shown that if a set of crossings is non-empty, then a lowest left to right crossing exists and we call it Π. We call $\mathcal{A}$ the set of left to right crossings of $\mathcal{B}(\ell)$.

For $\pi \in \mathcal{A}$ we denote by $(0, y_\pi)$ the last vertex where the path and the vertical axis intersect. For $\pi \in \mathcal{A}$ we write π_r for the part of π that connects $(0, y_\pi)$ to the right side of $\mathcal{B}(\ell)$. We also write π_r' for the reflection of π_r along the line $\{(\ell, k) : k \in \mathbb{Z}\}$. We write $\mathcal{A}_-$ for the set of paths π with $y_\pi \le 0$ and $\mathcal{A}_+$ for those with $y_\pi \ge 0$.

For a path $\pi \in \mathcal{A}$ we write V_π for the event that all the edges of π are open. We also write M_π for the event that there exists an open path from the top of $\mathcal{B}(\ell)' = [0, 2\ell] \times [-\ell, \ell]$ to $\pi_r \cup \pi_r'$ and M_π^- for the event that this path ends on π_r.

Let L^- be the event that there exists an open path in $\mathcal{A}_-$, resp. for L^+. Let L_π be the event that π is the lowest left to right open crossing of $\mathcal{B}(\ell)$.

Finally we write N for the event that there exists a left to right crossing of $\mathcal{B}(\ell)'$ and N^- for the event that this path starts from the negative part of the vertical axis.

It is clear from the picture that the event $\cup_{\pi \in \mathcal{A}_-} V_\pi \cap M_\pi^- \cap N^+$ implies the existence of an open left to right crossing of $\mathcal{B}(3/2\ell, \ell)$. Hence we get

$$\mathbb{P}_p(\mathrm{LR}(3/2\ell, \ell)) \ge \mathbb{P}_p\big(N^+ \cap \big(\cup_{\pi \in \mathcal{A}_-} (V_\pi \cap M_\pi^-)\big)\big) = \mathbb{P}_p\big(N^+ \cap G\big) .$$

We now observe that both events, N^+ and G, are increasing, and hence we can use the FKG inequality to obtain

$$\mathbb{P}_p(\mathrm{LR}(3/2\ell, \ell)) \ge \mathbb{P}_p\big(N^+\big)\, \mathbb{P}_p(G) .$$

By the square root trick we get

$$\mathbb{P}_p\big(N^+\big) \ge 1 - \sqrt{1 - \mathbb{P}_p(N^+ \cup N^-)} = 1 - \sqrt{1 - \alpha},$$

since $\mathbb{P}_p(N^+ \cup N^-) = \mathbb{P}_p(\mathrm{LR}(\ell)) = \alpha$ by assumption.

It remains to lower bound the probability of G. For this we have

$$\mathbb{P}_p(G) \ge \mathbb{P}_p\big(\cup_{\pi \in \mathcal{A}_-} (L_\pi \cap M_\pi^-)\big) = \sum_{\pi \in \mathcal{A}_-} \mathbb{P}_p\big(M_\pi^- \bigm| L_\pi\big)\, \mathbb{P}_p(L_\pi) ,$$

since the events appearing in the union are disjoint. We now claim that for all $\pi \in \mathcal{A}$ we have

$$\mathbb{P}_p\big(M_\pi^- \bigm| L_\pi\big) \ge 1 - \sqrt{1 - \alpha}. \tag{1.16}$$

Once we establish this, the rest of the proof follows easily. Indeed, substituting this into the bound for $\mathbb{P}_p(G)$ we get

$$\mathbb{P}_p(G) \ge \sum_{\pi \in \mathcal{A}_-} \mathbb{P}_p(L_\pi)\, (1 - \sqrt{1 - \alpha}).$$

But $\sum_{\pi\in\mathcal{A}_-}\mathbb{P}_p(L_\pi)=\mathbb{P}_p(L^-)$ and by the square root trick again we deduce

$$\mathbb{P}_p\big(L^+\big)\geq 1-\sqrt{1-\mathbb{P}_p(L^+\cup L^-)}=1-\sqrt{1-\alpha}.$$

So it remains to prove (1.16).

The idea is to say that the event that π is the lowest left to right crossing is not giving us more information than saying that all the edges of π are open. Indeed, what happens below π does not affect the occurrence of the event M_π. Therefore

$$\mathbb{P}_p\big(M_\pi^- \bigm| L_\pi\big) \text{``}=\text{''}\mathbb{P}_p\big(M_\pi^- \bigm| V_\pi\big)\,.$$

And now we can use the FKG inequality again since all events are increasing to get

$$\mathbb{P}_p\big(M_\pi^- \bigm| V_\pi\big)\geq \mathbb{P}_p\big(M_\pi^-\big)\geq 1-\sqrt{1-\mathbb{P}_p\big(M_\pi^+\cup M_\pi^-\big)}=1-\sqrt{1-\mathbb{P}_p(M_\pi)}\geq 1-\sqrt{1-\alpha},$$

since whenever there is a top to bottom crossing of $\mathcal{B}(\ell)'$ the event M_π happens. This now concludes the proof. □

1.9 Power law inequalities at the critical point

Theorem 1.42. *There exist finite positive constants $\alpha_1, A_1, \alpha_2, A_2, \alpha_3, A_3, \alpha_4, A_4$ so that for all $n \geq 1$ we have*

$$\frac{1}{2\sqrt{n}} \leq \mathbb{P}_{1/2}(0\leftrightarrow\partial\mathcal{B}_n)\leq A_1\cdot n^{-\alpha_1} \tag{1.17}$$

$$\frac{1}{2\sqrt{n}} \leq \mathbb{P}_{1/2}(|\mathcal{C}(0)|\geq n)\leq A_2\cdot n^{-\alpha_2} \tag{1.18}$$

$$\mathbb{E}[|\mathcal{C}(0)|^{\alpha_3}]<\infty. \tag{1.19}$$

Moreover, for all $p>1/2$ we have

$$\theta(p)\leq A_4\cdot\left(p-\frac{1}{2}\right)^{\alpha_4} \tag{1.20}$$

Remark 1.43. Recall that in Theorem 1.33, part (b) we proved that when $p>1/2$, then

$$\theta(p)\geq c\left(p-\frac{1}{2}\right),$$

where c is a positive constant.

The following easy application of RSW will be a crucial ingredient of the proof.

Lemma 1.44. *Let $O(\ell)$ be the event that there exists an open circuit in the annulus $A(\ell)=\mathcal{B}(3\ell)\setminus\mathcal{B}(\ell)$ containing 0 in its interior. Then there exists a positive constant ζ such that for all $\ell\geq 1$ we have*

$$\mathbb{P}_{1/2}(O(\ell))\geq\zeta.$$

Proof. Let A_n be the event that the left side of $[0, n+1] \times [0, n]$ is joined to the right side along an open path.

Then in (1.12) in the proof of Theorem 1.37 we showed using duality that

$$\mathbb{P}_{1/2}(A_n) = \frac{1}{2}.$$

It is obvious that this left to right crossing also contains a left to right crossing of the box $[0, n] \times [0, n]$. Therefore, we showed that $\mathbb{P}_{1/2}(\mathrm{LR}(\ell)) \geq 1/2$ for all $\ell \geq 1$.

Hence, from Theorem 1.38 (RSW) we obtain that for all $\ell \geq 1$

$$\mathbb{P}_{1/2}(O(\ell)) \geq 2^{-24} \left(1 - \frac{1}{2}\sqrt{3}\right)^{48}$$

and this completes the proof. □

Proof of Theorem 1.42. The left hand side of (1.17) is left as an exercise. We turn now to the right hand side.

We work on the dual lattice again and write $\mathcal{B}(k)_d = \mathcal{B}(k) + (1/2, 1/2)$ for the dual box, $A(\ell)_d$ for the dual annulus $A(\ell)_d = \mathcal{B}(3\ell)_d \setminus \mathcal{B}(\ell)_d$. Let $O(\ell)_d$ be the event that there exists a closed dual circuit in $A(\ell)_d$ containing the dual origin $(1/2, 1/2)$ in its interior. Duality and Lemma 1.44 immediately give

$$\mathbb{P}_{1/2}(O(\ell)_d) \geq \zeta, \quad \forall\, \ell \geq 1.$$

For all $k \geq 0$ we now get

$$\mathbb{P}_{1/2}\Big(0 \leftrightarrow \partial\mathcal{B}(3^k + 1)\Big) \leq \mathbb{P}_{1/2}(O(3^r)_d \text{ does not occur } \forall\, r < k)\,.$$

But notice that the annuli $A(3^r)_d$ for different values of r are disjoint, and hence the events appearing on the right hand side above are independent. Therefore, we obtain

$$\mathbb{P}_{1/2}\Big(0 \leftrightarrow \partial\mathcal{B}(3^k + 1)\Big) \leq (1 - \zeta)^k.$$

For every $n \geq 1$ we can find k such that $3^k + 1 \leq n < 3^{k+1} + 1$. We thus deduce

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial\mathcal{B}(n)) \leq \mathbb{P}_{1/2}\Big(0 \leftrightarrow \partial\mathcal{B}(3^k + 1)\Big) \leq (1 - \zeta)^k \leq A_1 \cdot n^{-\alpha_1},$$

where A_1 and α_1 are positive constants. This proves (1.17).

The second inequality follows directly from (1.17). For (1.19) we have

$$\mathbb{E}[|\mathcal{C}_0|^\gamma] = \sum_{n=0}^{\infty} \mathbb{P}_{1/2}(|\mathcal{C}_0|^\gamma \geq n) < \infty$$

by choosing a suitable γ as a function of α_2.

We now prove (1.20). First of all we note that for all $n \geq 1$

$$\theta(p) \leq \mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n)\,.$$

Take now $p > 1/2$. Since there are at most $18n^2$ edges in the box $\mathcal{B}_n$, from Corollary 1.32 we get

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial\mathcal{B}_n) \geq \left(\frac{1}{2p}\right)^{18n^2} \cdot \mathbb{P}_p(0 \leftrightarrow \partial\mathcal{B}_n)\,.$$

This now gives us

$$\theta(p) \leq (2p)^{18n^2} \cdot \mathbb{P}_{1/2}(0 \leftrightarrow \partial\mathcal{B}_n) \leq (2p)^{18n^2} \cdot A_1 \cdot n^{-\alpha_1},$$

where for the last inequality we used (1.17). Taking now $n = [(\log(2p))^{-1/2}]$ and noting that $n \sim (2p-1)^{-1/2}$ as $p \downarrow 1/2$ yields

$$\theta(p) \leq A_1 \cdot e^{18} \cdot n^{-\alpha_1} \leq A_4 \cdot \left(p - \frac{1}{2}\right)^{\alpha_4}$$

and this concludes the proof. $\square$

1.10 Grimmett Marstrand theorem

For any connected graph G we write $p_c(G)$ for the critical probability of bond percolation on G. We now define for each $k \geq 1$ the slab of thickness k by

$$S_k = \mathbb{Z}^2 \times \{0, 1, 2, \ldots, k\}^{d-2}.$$

We write $p_c(S_k)$ for the critical probability of S_k. Since $S_k \subseteq S_{k+1} \subseteq \mathbb{Z}^d$, we immediately get

$$p_c(S_k) \geq p_c(S_{k+1}) \geq p_c(\mathbb{Z}^d).$$

Therefore, the limit $\lim_{k\to\infty} p_c(S_k)$ exists and we call it p_c^{slab} which satisfies $p_c^{\text{slab}} \geq p_c$.

It was a big open question for many years to prove that $p_c^{\text{slab}} = p_c$.

This was finally proved by Grimmett and Marstrand and it is one of the most important theorems in the study of the supercritical regime for $d \geq 3$. For $d = 2$, the supercritical regime corresponds to subcritical for the dual process, but in higher dimensions the picture is different.

A lot of theorems can be proved under the assumption that $p > p_c^{\text{slab}}$.

Theorem 1.45 (Grimmett and Marstrand (1990))**.**
(a) *Let $d \geq 2$ and let F be an infinite connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$. For each $\eta > 0$ there exists an integer k such that*

$$p_c(2kF + \mathcal{B}_k) \leq p_c + \eta.$$

(b) *If $d \geq 3$, then $p_c^{\text{slab}} = p_c$.*

Proof that (a)$\Rightarrow$ (b) Suppose that $d \geq 3$. Taking $F = \mathbb{Z}^2 \times \{0\}^{d-2}$ we have that

$$2kF + \mathcal{B}_k = \mathbb{Z}^2 \times ([-k, k]^{d-2} \cap \mathbb{Z}^{d-2}),$$

which is a translate of S_{2k}. Therefore

$$p_c(S_{2k}) = p_c(2kF + \mathcal{B}_k) \to p_c \text{ as } k \to \infty,$$

which then gives that $p_c = p_c^{\mathrm{slab}}$. $\square$

An application of Theorem 1.45 is the following theorem on the decay of the probability that 0 is connected to distance n and belongs to a finite component when $p > p_c$. For $d = 2$ this can be proved using duality, but for $d \geq 3$ it can be established for all $p > p_c^{\mathrm{slab}}$.

Theorem 1.46. *Let $d \geq 3$ and $p > p_c$. There exists a positive constant c so that for all $n \geq 1$ we have*

$$\mathbb{P}_p(0 \leftrightarrow \partial \mathcal{B}_n, |\mathcal{C}(0)| < \infty) \leq e^{-cn}.$$

1.11 Conformal invariance of crossing probabilities $p = p_c$

We follow [4].

In this section we consider site percolation on the triangular lattice $\mathbb{T}$. Each vertex is painted black or white with probability p and $1 - p$ respectively independently over different vertices.

It may be proved similarly to $\mathbb{Z}^2$ that $p_c(\mathbb{T}) = 1/2$.

Let us introduce some notation and definitions.

Let $D \subseteq \mathbb{C}$ be an open simply connected domain in $\mathbb{R}^2$. We assume that its boundary ∂D is a Jordan curve. Let a, b, c be distinct points of ∂D taken in anticlockwise order around ∂D. The Riemann mapping theorem guarantees the existence of a conformal map φ from D to the interior of the equilateral triangle T of $\mathbb{C}$ with vertices $A = 0$, $B = 1$ and $C = e^{\pi i/3}$. Such a φ can be extended to the boundary ∂D in such a way that it becomes a homeomorphism from $D \cup \partial D$ to the closed triangle T. There exists a unique conformal map φ that maps a, b, c to A, B, C respectively. If $x \in \partial D$ lying on the arc bc, then under the conformal map $X = \varphi(x)$ lies on the arc BC of T.

We now rescale the triangular lattice $\mathbb{T}$ to have mesh size δ and we are interested in the probability $\mathbb{P}_\delta(ac \leftrightarrow bx \text{ in } D)$ of having an open path joining ac to bx in the domain D. The RSW estimates also hold for the triangular lattice, and hence the probability above can be shown to be uniformly bounded away from 0 and 1 as $\delta \to 0$. So it seems reasonable that this probability should converge as $\delta \to 0$. Cardy's formula tells us the values of this limit.

Theorem 1.47 (Cardy's formula)**.** *We have*

$$\mathbb{P}_\delta(ac \leftrightarrow bx \ \textit{in} \ D) \to |BX| \ \textit{as} \ \delta \to 0.$$

Cardy stated the limit of the probability above in terms of a hypergeometric function of a certain cross-ratio. To derive it he used arguments from conformal field theory. Lennart Carlesson recognised the hypergeometric function in terms of the conformal map from a rectangle to a triangle and conjectured the simple form of the theorem above. The theorem was then proved by Smirnov in 2001.

Theorem 1.48 (Smirnov, 2001)**.** *Cardy's formula holds for the triangular lattice.*

If two domains (Ω, a, b, c, d) and $(\Omega', a', b', c', d')$ are conformally equivalent (i.e. there exists a conformal map from one to the other one), then

$$\mathbb{P}(ac \leftrightarrow bd \ \textit{in} \ \Omega) = \mathbb{P}\big(a'c' \leftrightarrow b'd' \ \textit{in} \ \Omega'\big)\,.$$

We now introduce the exploration process – dynamical way of discovering the configuration in a discrete domain Ω, i.e. made of small hexagons. We use the following convenient way of representing site percolation on $\mathbb{T}$: we colour each face of the dual hexagonal lattice $\mathbb{H}$ in the same colour as the corresponding site in $\mathbb{T}$. Hence we can think of site percolation on $\mathbb{T}$ as a random colouring of the faces of $\mathbb{H}$.

Colour the positive half line black and the negative one white. Then one can define the exploration process by always keeping a black hexagon on the right and a white one on the left. If we rescale the mesh size of the hexagonal lattice, what is the scaling limit of this exploration curve?

It turns out that the limit is an SLE(6) curve. The proof of this uses the *locality property* of percolation: what you have not explored so far is independent of what has been explored. The same property holds for SLE(6). In particular both processes have the same hitting probabilities. For the exploration process, one can get these hitting probabilities using Cardy's formula.

Let us take a continuous curve γ_t in the upper half plane $\mathbb{H}$. Let K_t be such that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$. Then there is a conformal map $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ satisfying

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - a(t)},$$

where $a(t)$ is a real-valued function of t and it is exactly the point where the tip of the curve is mapped via g_t, i.e. $g_t(\gamma_t) = a(t)$. Also we have $g_0(z) = z$.

In order for this curve to describe the scaling limit of many discrete structures, the function a must have the following properties:

(1) a must be continuous

(2) it must have i.i.d. increments

(3) a must be symmetric, i.e. a has the same law as $-a$.

There is only one type of function with all these properties, namely any multiple of a standard Brownian motion. So taking $a(t) = \sqrt{\kappa} B_t$ gives us the SLE(κ) curve.

More on SLE next term!

2 Random walks on graphs

In this section we are following closely [5, Chapter 9].

Let $G = (V, E)$ be a finite connected graph with set of vertices V and set of edges E. We endow it with non-negative numbers $(c(e))_{e\in E}$ that we call conductances. We write $c(x, y) = c(\{x, y\})$ and clearly $c(x, y) = c(y, x)$. The reciprocal $r(e) = 1/c(e)$ is called the resistance of the edge e.

We now consider the Markov chain on the nodes of G with transition matrix

$$P(x, y) = \frac{c(x, y)}{c(x)},$$

where $c(x) = \sum_{y:y\sim x} c(x, y)$. This process is called the weighted random walk on G with edge weights $(c(e))_e$.

This process is reversible with respect to $\pi(x) = c(x)/c_G$, where $c_G = \sum_{x\in V} c(x)$, since

$$\pi(x)P(x, y) = \frac{c(x)}{c_G} \cdot \frac{c(x, y)}{c(x)} = \frac{c(x, y)}{c_G} = \pi(y)P(y, x)$$

and π is stationary for P, i.e. $\pi P = \pi$.

When $c(e) = 1$ for all edges e, we call the Markov chain with transition matrix P a simple random walk on G. In this case

$$P(x, y) = \begin{cases} \frac{1}{d(x)} & \text{if } y \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to show that every reversible Markov chain is a weighted random walk on a graph. Indeed, suppose that P is a transition matrix which is reversible with respect to the stationary distribution π. Then we declare $\{x, y\}$ an edge if $P(x, y) > 0$. Reversibility implies that $P(y, x) > 0$ if $P(x, y) > 0$, and hence this is well-defined. We define conductances on the edges by setting $c(x, y) = \pi(x)P(x, y)$. Again by reversibility this is symmetric and with this choice of weights we get $\pi(x) = c(x)$. The study of reversible Markov chains is thus equivalent to the study of random walks on weighted graphs.

Let now P be a transition matrix which is irreducible with state space Ω. (We do not assume that P is reversible.)

A function $h : \Omega \to \mathbb{R}$ is called harmonic for P at the vertex x if

$$h(x) = \sum_{y\in\Omega} P(x, y)h(y).$$

For a subset $B \subseteq \Omega$ we define the hitting time of B by

$$\tau_B = \min\{t \geq 0 : X_t \in B\}.$$

Proposition 2.1. *Let X be an irreducible Markov chain with transition matrix P and let $B \subseteq \Omega$. Let $f : B \to \mathbb{R}$ be a function defined on B. Then the function $h(x) = \mathbb{E}_x[f(X_{\tau_B})]$ is the unique extension $h : \Omega \to \mathbb{R}$ of f such that $h(x) = f(x)$ for all $x \in B$ and h is harmonic for P at all $x \in \Omega \setminus B$.*

Proof. Clearly $h(x) = f(x)$ for all $x \in B$. By conditioning on the first step of the Markov chain we get for $x \notin B$

$$h(x) = \sum_{y \in \Omega} P(x,y)\mathbb{E}_x[f(X_{\tau_B}) \mid X_1 = y] = \sum_y P(x,y)\mathbb{E}_y[f(X_{\tau_B})] = \sum_y P(x,y)h(y),$$

where for the second equality we used the Markov property.

We now turn to show uniqueness. Let h' be another function satisfying the same conditions as h. Then the function $g = h - h'$ is harmonic on $\Omega \setminus B$ and $g = 0$ on B. We first show that $g \leq 0$. Assume the contrary and let $x \notin B$ belong to the set

$$A = \left\{ x : \, g(x) = \max_{y \in \Omega \setminus B} g(y) \right\}.$$

Suppose that $P(x,y) > 0$. If $g(y) < g(x)$, then harmonicity of g on $\Omega \setminus B$ implies that

$$g(x) = \sum_{z \in \Omega} g(z)P(x,z) = g(y)P(x,y) + \sum_{z \neq y} P(x,z)g(z) < \max_{y \in \Omega \setminus B} g(y),$$

which is clearly a contradiction. Hence it follows that $g(y) = \max_z g(z)$, which means that $y \in A$. By irreducibility we continue in the same way and we finally get that $g(x) = 0$, since we eventually get to the boundary B. Hence this proves that $\max g = 0$. Similarly, we can prove that $\min g = 0$. So we deduce that $g = 0$. □

2.1 Electrical networks

The goal of this section is to explain the connection between random walks and electrical networks.

Again here $G = (V, E)$ will be a finite connected graph with conductances $(c(e))_e$ on the edges. We distinguish two vertices a and b that will be the source and the sink respectively.

Definition 2.2. A flow θ on G is a function defined on oriented edges $\vec{e} = (x,y)$ of E satisfying

$$\theta(x,y) = -\theta(y,x).$$

The **divergence** of the flow θ is defined to be $\operatorname{div}\theta(x) = \sum_{y \sim x} \theta(x,y)$.

We note that by the antisymmetric property of θ we get

$$\sum_x \operatorname{div}\theta(x) = \sum_{(x,y) \in E} (\theta(x,y) + \theta(y,x)) = 0.$$

Definition 2.3. A **flow θ from a to b** is a flow such that

- $\operatorname{div}\theta(x) = 0 \quad \forall\, x \notin \{a,b\}$ ("flow in equals flow out" – Kirchoff's node law)
- $\operatorname{div}\theta(a) \geq 0$.

The **strength** of a flow θ from a to b is defined to be $\|\theta\| := \operatorname{div}\theta(a)$. A **unit flow** is a flow with $\|\theta\| = 1$.

We note that $\operatorname{div}\theta(b) = -\operatorname{div}\theta(a)$.

A **voltage** W is a harmonic function on $V \setminus \{a, b\}$. By Proposition 2.1 a voltage always exists and is uniquely determined by its boundary values $W(a)$ and $W(b)$.

Given a voltage W on G we define the **current flow** I on oriented edges via

$$I(x, y) = \frac{W(x) - W(y)}{r(x, y)} = c(x, y)(W(x) - W(y)).$$

Exercise: Check it is a flow and then prove that the unit current flow is unique.

By the definition we immediately see that the current flow satisfies Ohm's law:

$$r(x, y)I(x, y) = W(x) - W(y).$$

The current flow also satisfies the **cycle law**: if the oriented edges $\vec{e}_1, \ldots, \vec{e}_n$ form an oriented cycle, then

$$\sum_{i=1}^{n} r(\vec{e}_i)I(\vec{e}_i) = 0.$$

Proposition 2.4. *Let I be a current and θ a flow from a to z satisfying the cycle law for any cycle. If $\|\theta\| = \|I\|$, then $\theta = I$.*

Proof. Consider the function $f = \theta - I$. Then since θ and I are flows with the same strength, it follows that f satisfies Kirchoff's node law at all nodes and the cycle law. Suppose that $\theta \neq I$. Then without loss of generality, there must exist an edge $\vec{e}_1$ such that $f(\vec{e}_1) > 0$. Since $\sum_{y \sim x} f(x, y) = 0$, we get that there must exist an edge $\vec{e}_2$ to which $\vec{e}_1$ leads such that $f(\vec{e}_2) > 0$. Continuing in this way we get a sequence of oriented edges with positive value of f. Since the graph is finite, at some point this sequence of edges should revisit a point. This then violates the cycle law. Hence $\theta = I$. $\square$

2.2 Effective resistance

Let W_0 be the voltage with $W_0(a) = 1$ and $W_0(z) = 0$. By the uniqueness of harmonic functions, we obtain that any other voltage W is given by

$$W(x) = (W(a) - W(z))W_0(x) + W(z).$$

We call I_0 the current flow associated with W_0. Note that by the definition of the current flow, its strength is given by

$$\|I\| = \sum_{x \sim a} \frac{W(a) - W(x)}{r(a, x)} = (W(a) - W(z)) \|I_0\|.$$

We thus see that the ratio

$$\frac{W(a) - W(z)}{\|I\|}$$

is independent of W. We define this to be the effective resistance

$$R_{\text{eff}}(a, z) := \frac{W(a) - W(z)}{\|I\|}$$

and the reciprocal is called the effective conductance, $C_{\text{eff}}(a, z)$.

Why is it called effective resistance? Suppose that we wanted to replace the whole network by single edge joining a and z with resistance $R_{\text{eff}}(a, z)$. Then if we apply the same voltage at a and z in both networks, then the same amount of current would flow through.

We are now ready to state the connection between random walks and electrical networks.

We define $\tau_x^+ = \min\{t \geq 1 : X_t = x\}$.

Proposition 2.5. *Let X be a reversible chain on a finite state space. For any $x, a, z \in \Omega$ we have*

$$\mathbb{P}_a\big(\tau_z < \tau_a^+\big) = \frac{1}{c(a)R_{\text{eff}}(a, z)}.$$

Proof. The function $f(x) = \mathbb{P}_x(\tau_z < \tau_a)$ is a harmonic function on $\Omega \setminus \{a, z\}$ and $f(a) = 0$ and $f(z) = 1$. So from Proposition 2.1 we get that f has to be equal to the function

$$h(x) = \frac{W(a) - W(x)}{W(a) - W(z)},$$

where W is a voltage, since they are both harmonic with the same boundary values. Therefore, we obtain

$$\mathbb{P}_a\big(\tau_z < \tau_a^+\big) = \sum_x P(a, x)\mathbb{P}_x(\tau_z < \tau_a) = \sum_{x \sim a} \frac{c(a, x)}{c(a)} \cdot \frac{W(a) - W(x)}{W(a) - W(z)}.$$

By the definition of the current flow, the above sum is equal to

$$\frac{\sum_{x \sim a} I(a, x)}{c(a)(W(a) - W(z))} = \frac{1}{c(a)R_{\text{eff}}(a, z)}$$

and this proves the proposition. □

Definition 2.6. The **Green's function** for a random walk stopped at a stopping time τ is defined to be

$$G_\tau(a, x) := \mathbb{E}_a\left[\sum_{t=0}^{\infty} \mathbf{1}(X_t = x, \tau > t)\right].$$

Lemma 2.7. *Let X be a reversible Markov chain. Then for all a, z we have*

$$G_{\tau_z}(a, a) = c(a)R_{\text{eff}}(a, z).$$

Proof. The number of visits to a before the first hitting time of z has a geometric distribution with parameter $\mathbb{P}_a(\tau_z < \tau_a^+)$. The statement follows from Proposition 2.5. □

There are some ways of simplifying a network without changing quantities of interest.

Conductances in parallel add: let e_1 and e_2 be edges sharing the same endvertices. Then we can replace both edges by a single edge of conductance equal to the sum of the conductances. Then the same current will flow through and the same voltage difference will be applied. To see it, check Kirchoff's and Ohm's laws with $I(\vec{e}) = I(\vec{e}_1) + I(\vec{e}_2)$.

Resistances in series add: if $v \in V \setminus \{a, z\}$ is a node of degree 2 with neighbours v_1 and v_2, we could replace the edges (v, v_1) and (v, v_2) by a single edge (v_1, v_2) of resistance $r(v_1, v_2) = r(v, v_1) + r(v, v_2)$. To see it, check Kirchoff's and Ohm's laws with $I(v_1, v_2) = I(v_1, v) = I(v, v_2)$ and the same as before everywhere else.

Gluing: If two vertices have the same voltage, we can glue them to a single vertex, while keeping all existing edges. Since current never flows between vertices with the same voltage, potentials and currents remain unchanged.

Example 2.8. Let a and b be two vertices on a finite connected tree T. Then the effective resistance $R_{\text{eff}}(a, b)$ is equal to the distance on the tree between a and b.

Definition 2.9. Let θ be a flow on a finite connected graph G. We define its energy by

$$\mathcal{E}(\theta) = \sum_e (\theta(e))^2 r(e),$$

where the sum is taken over all unoriented edges e. (Note that since θ is antisymmetric, we did not need to take direction on e.)

The following theorem gives an equivalent definition of effective resistance as the minimal energy over all flows of unit strength from a to z.

Theorem 2.10 (Thomson's principle)**.** *Let G be a finite connected graph with edge conductances $(c(e))_e$. For all a and z we have*

$$R_{\text{eff}}(a, z) = \inf\{\mathcal{E}(\theta) : \theta \textit{ is a unit flow from } a \textit{ to } z\}.$$

The unique minimiser above is the unit current flow from a to z.

Proof. We follow [4].

Let i be the unit current flow from a to z associated to the potential φ.

We start by showing that

$$R_{\text{eff}}(a, z) = \mathcal{E}(i).$$

Using that i is a flow from a to z and Ohm's law we have

$$\mathcal{E}(i) = \frac{1}{2} \sum_{\substack{u,v \\ u \sim v}} i(u, v)^2 r(u, v) = \frac{1}{2} \sum_{\substack{u,v \\ u \sim v}} i(u, v)(\varphi(u) - \varphi(v)) = \varphi(a) - \varphi(z) = R_{\text{eff}}(a, z).$$

Let j be another flow from a to z of unit strength. The goal is to show that $\mathcal{E}(j) \geq \mathcal{E}(i)$.

We define $k = j - i$. Then this is a flow of 0 strength. So we now get

$$\begin{aligned}
\mathcal{E}(j) &= \sum_e (j(e))^2 r(e) = \sum_e (i(e) + k(e))^2 r(e) \\
&= \sum_e (i(e))^2 r(e) + \sum_e (k(e))^2 r(e) + 2 \sum_e k(e) i(e) r(e) \\
&= \mathcal{E}(i) + \mathcal{E}(k) + 2 \sum_e k(e) i(e) r(e).
\end{aligned}$$

We now show that

$$\sum_e k(e)i(e)r(e) = 0.$$

Since i is the unit current flow associated with φ, for $e = (x, y)$ it satisfies

$$i(x,y) = \frac{\varphi(x) - \varphi(y)}{r(x,y)}.$$

Substituting this above we obtain

$$\sum_e k(e)i(e)r(e) = \frac{1}{2} \cdot \sum_x \sum_{y \sim x} (\varphi(x) - \varphi(y))k(x,y) = \frac{1}{2} \cdot \sum_x \sum_{y \sim x} \varphi(x)k(x,y) + \frac{1}{2} \cdot \sum_x \sum_{y \sim x} \varphi(x)k(x,y),$$

where for the last equality we used the antisymmetric property of k. Since k is a flow of 0 strength, we get that both these sums are equal to 0. Therefore this proves that

$$\mathcal{E}(j) \geq \mathcal{E}(i)$$

with equality if and only if $\mathcal{E}(k) = 0$ which is equivalent to $k = 0$. $\square$

Theorem 2.11 (Rayleigh monotonicity principle)**.** *The effective resistance is a monotone increasing function as a function of the component resistances, i.e. if $(r(e))_e$ and $(r'(e))_e$ satisfy $r(e) \leq r'(e)$ for all e, then*

$$R_{\text{eff}}(a, z; r) \leq R_{\text{eff}}(a, z; r').$$

Proof. Let i and i' be the unit current flows associated to the resistances $r(e)$ and $r'(e)$ respectively. Then by Thomson's principle we get

$$R_{\text{eff}}(a, z; r) = \sum_e (i(e))^2 r(e) \leq \sum_e (i'(e))^2 r(e),$$

where the inequality follows, since the energy is minimised by the unit current flow i. Using the assumption on the resistances we now obtain

$$\sum_e (i'(e))^2 r(e) \leq \sum_e (i'(e))^2 r'(e) = R_{\text{eff}}(a, z; r')$$

and this concludes the proof. $\square$

Corollary 2.12. *Let G be a finite connected graph. Suppose we add an edge which is not adjacent to a. Then this increases the escape probability $\mathbb{P}_a(\tau_z < \tau_a^+)$.*

Proof. Recall from Proposition 2.5 that

$$\mathbb{P}_a\big(\tau_z < \tau_a^+\big) = \frac{1}{c(a) R_{\text{eff}}(a, z)}.$$

Adding an edge means that we decrease the resistance of the edge from ∞ to a finite number. Hence from Rayleigh's monotonicity principle we get that the effective resistance will decrease. $\square$

Corollary 2.13. *The operation of gluing vertices together cannot increase the effective resistance.*

We now present a nice technique due to Nash and Williams to obtain lower bounds on effective resistances.

Definition 2.14. We call a set of edges Π an edge–cutset separating a from z if every path from a to z uses an edge of Π.

Proposition 2.15 (Nash-Williams inequality)**.** *If (Π_k) are disjoint edge-cutsets which separate a from z, then*

$$R_{\text{eff}}(a,z) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Proof. By Thomson's principle it suffices to prove that for any flow θ from a to z of unit strength we have

$$\sum_e (\theta(e))^2 r(e) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Since the sets Π_k are disjoint, we get

$$\sum_e (\theta(e))^2 r(e) \geq \sum_k \sum_{e \in \Pi_k} (\theta(e))^2 r(e).$$

By the Cauchy-Schwarz inequality we now get

$$\left(\sum_{e \in \Pi_k} c(e) \right) \cdot \left(\sum_{e \in \Pi_k} r(e)(\theta(e))^2 \right) \geq \left(\sum_{e \in \Pi_k} \sqrt{c(e)}\sqrt{r(e)}|\theta(e)| \right)^2 = \left(\sum_{e \in \Pi_k} |\theta(e)| \right)^2.$$

But since the sets Π_k are cutsets separating a from z and θ has unit strength, this last sum is at least 1. Rearranging completes the proof. $\square$

Proposition 2.16. *Let $a = (1,1)$ and $z = (n,n)$ be the opposite corners of the box $B_n = [1,n]^2 \cap \mathbb{Z}^2$. Then*

$$R_{\text{eff}}(a,z) \geq \frac{\log(n-1)}{2}.$$

Remark 2.17. The effective resistance between a and z is also upper bounded by $\log n$. To prove it one defines a flow and shows that its energy is at most $\log n$. For more details see [5, Proposition 9.5].

Proof of Proposition 2.16. Let $\Pi_k = \{(v,u) \in B_n : \|v\|_\infty = k \text{ and } \|u\|_\infty = k-1\}$. Then (Π_k) are disjoint edge-cutsets separating a from z. By counting the number of edges we get $|\Pi_k| = 2(k-1)$ and since $c(e) = 1$ for all edges e we get

$$R_{\text{eff}}(a,z) \geq \sum_{k=2}^{n} \frac{1}{2(k-1)} \geq \frac{\log(n-1)}{2}$$

and this completes the proof. $\square$

Lemma 2.18. *Let X be an irreducible Markov chain on a finite state space. Let τ be a stopping time satisfying $\mathbb{E}[\tau] < \infty$ and $\mathbb{P}_a(X_\tau = a) = 1$ for some a in the state space. Then for all x we have*

$$G_\tau(a, x) = \mathbb{E}_a[\tau] \cdot \pi(x).$$

Proposition 2.19 (Commute time identity)**.** *Let X be a reversible Markov chain on a finite state space. Then for all a, b we have*

$$\mathbb{E}_a[\tau_{a,b}] = \mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = c(G) \cdot R_{\text{eff}}(a, z),$$

where $c(G) = 2\sum_e c(e)$.

Proof. The stopping time $\tau_{a,b}$ which is the first time the walk comes back to a after having visited b satisfies $\mathbb{P}_a\big(X_{\tau_{a,b}} = a\big) = 1$. Also note that we only visit a before time τ_b, i.e.

$$G_{\tau_{a,b}}(a, a) = G_{\tau_b}(a, a) = c(a)R_{\text{eff}}(a, b),$$

where the last equality follows from Proposition 2.5. We can now apply the previous lemma to finish the proof. □

Remark 2.20. The previous identity immediately gives us that the effective resistance satisfies the triangle inequality. Hence the effective resistance defines a metric on the graph G (the other two properties are trivially satisfied).

2.3 Transience vs recurrence

So far we have been focusing on finite state spaces. In this section we will see how we can use the electrical network point of view to determine transience and recurrence properties of graphs.

Let $G = (V, E)$ be a countable graph and let 0 be a distinguished point. Let $G_k = (V_k, E_k)$ be an exhaustion of G by finite graphs, i.e. $V_n \subseteq V_{n+1}$ and E_n contains all edges of E with endpoints in V_n for all n, $0 \in V_n$ for all n and $\cup_n V_n = V$.

For every n we construct a graph G_n^* by gluing all the points of $V \setminus V_n$ into a single point z_n and setting $c(x, z_n) = \sum_{z \in V\setminus V_n} c(x, z)$ and define

$$R_{\text{eff}}(0, \infty) = \lim_{n\to\infty} R_{\text{eff}}(0, z_n; G_n^*).$$

Check by Rayleigh's monotonicity principle that this limit does exist and is independent of the choice of exhaustion. We thus have

$$\mathbb{P}_0\big(\tau_0^+ = \infty\big) = \lim_{n\to\infty} \mathbb{P}_a\big(\tau_{z_n} < \tau_0^+\big) = \lim_{n\to\infty} \frac{1}{c(0)R_{\text{eff}}(0, z_n; G_n^*)} = \frac{1}{c(0)R_{\text{eff}}(0, \infty)}. \tag{2.1}$$

We can define a flow from 0 to ∞ on an infinite graph as an antisymmetric function on the edges with $\text{div}\theta(\text{x}) = 0$ for all $x \neq 0$.

Proposition 2.21. *Let G be a countable connected weighted graph with conductances $(c(e))_e$ and let 0 be a distinguished vertex.*

(a) *A random walk on G is recurrent if and only if $R_{\text{eff}}(0,\infty)=\infty$.*

(b) *A random walk on G is transient if and only if there exists a unit flow i from 0 to ∞ of finite energy $\mathcal{E}(i)=\sum_e (i(e))^2 r(e)<\infty$.*

Proof. (a) The first part follows directly from (2.1). If the walk is recurrent, then

$$\mathbb{P}_0\big(\tau_0^+=\infty\big)=0,$$

and hence $R_{\text{eff}}(0,\infty)=\infty$ and vice versa.

(b) For the second part let G_n be an exhaustion of G by finite graphs. Then by definition

$$R_{\text{eff}}(0,\infty)=\lim_{n\to\infty} R_{\text{eff}}(0,z_n).$$

Let i_n be the unit current flow from 0 to z_n on the graph G_n and let v_n be the corresponding voltage. Then by Thomson's principle we get

$$R_{\text{eff}}(0,z_n)=\mathcal{E}(i_n).$$

Suppose now that there exists a unit flow θ from 0 to ∞ of finite energy. We call θ_n the restriction of θ to the graph G_n. Then θ_n is a unit flow from a to z_n in G_n. Applying Thomson's principle we obtain

$$\mathcal{E}(i_n)\le\mathcal{E}(\theta_n)\le\mathcal{E}(\theta)<\infty.$$

Therefore we get

$$R_{\text{eff}}(0,\infty)=\lim_{n\to\infty}\mathcal{E}(i_n)<\infty,$$

which implies that the walk is transient from the first part.

Suppose now that the walk is transient. Then $R_{\text{eff}}(0,\infty)<\infty$ from the first part. We want to construct a unit flow from 0 to ∞ of finite energy. Since $R_{\text{eff}}(0,\infty)<\infty$ we get that $\lim_{n\to\infty}\mathcal{E}(i_n)<\infty$, so there exists $M>0$ such that $\mathcal{E}(i_n)\le M$ for all n.

We now start a random walk from 0 and call $Y_n(x)$ the number of visits to x up until it hits z_n. We also call $Y(x)$ the total number of visits to x. It is clear that $Y_n(x)\uparrow Y(x)$ as $n\to\infty$, and hence by monotone convergence we get

$$\lim_{n\to\infty}\mathbb{E}_0[Y_n(x)]\uparrow\mathbb{E}_0[Y(x)]\,.$$

Since the walk is assumed to be transient, we have $\mathbb{E}_0[Y(x)]<\infty$. It is not hard to check that the function $G_{\tau_{z_n}}(0,x)/c(x)$ is a harmonic function with value 0 at z_n, and it is equal to $v_n(x)$. (Use reversibility and same proof as in Proposition 2.1). So we get

$$\lim_{n\to\infty} c(x)v_n(x)=c(x)v(x)$$

for some function v which is finite. Therefore, we can define

$$i(x,y)=c(x,y)(v(x)-v(y))=\lim_{n\to\infty} c(x,y)(v_n(x)-v_n(y)).$$

Since $\mathcal{E}(i_n) \le M$ for all n using dominated convergence one can show that i is a unit flow from 0 to ∞ of finite energy (Check!). $\square$

Corollary 2.22. *If $G \subseteq G'$ and G' is recurrent, then G is also recurrent. If G is transient, then G' is also transient.*

Corollary 2.23. *Simple random walk is recurrent on $\mathbb{Z}^2$ and transient on $\mathbb{Z}^d$ for all $d \ge 3$.*

Proof. For $d = 2$ we construct a new graph in which for each k we identify all vertices at distance k from 0. By the series/parallel law we see that

$$R_{\text{eff}}(0, \partial\Lambda_n) \ge \sum_{i=1}^{n-1} \frac{1}{8i-4}.$$

Therefore, we get that

$$R_{\text{eff}}(0, n) \ge c \log n \to \infty \quad \text{ as } n \to \infty.$$

For $d = 3$ we are going to construct a flow of finite energy. To each directed edge of $\mathbb{Z}^3$ we attach an orthogonal unit square intersecting e at its midpoint m_e. We now define the absolute value of $\theta(\vec{e})$ to be the area of the radial projection of this square onto the sphere of radius $1/4$ centred at the origin. We take $\theta(\vec{e})$ with positive sign if $\vec{e}$ points in the same direction as the radial vector from 0 to m_e and negative otherwise. By considering the projections of all faces of the unit cube centred at a lattice point, we can see that θ satisfies Kirchoff's node law at all vertices except for 0 (Check!). Hence θ is a flow from 0 to ∞ in $\mathbb{Z}^3$. Its energy is given by

$$\mathcal{E}(\theta) \le \sum_n c_1 n^2 \cdot \left(\frac{c_2}{n^2}\right)^2 < \infty,$$

and hence this proves transience.

An alternative proof goes via embedding a binary tree with resistance between edges from level $n-1$ to n equal to ρ^n for a suitable $\rho > 0$. Then the effective resistance of this tree is given by

$$R_{\text{eff}}(0, \infty) = \sum_{n=1}^{\infty} \left(\frac{\rho}{2}\right)^n.$$

Taking $\rho < 2$ makes it finite.

We now want to embed this tree in $\mathbb{Z}^3$ in such a way that a vertex at distance $n-1$ and a neighbour at distance n are separated by a path of length ρ^n.

The surface of a ball of radius k in $\mathbb{R}^3$ is of order k^2, so in order to be able to accommodate this tree in $\mathbb{Z}^3$ we need

$$(\rho^n)^2 \ge 2^n,$$

which then gives $\rho > \sqrt{2}$. This then gives that the effective resistance from 0 to ∞ in $\mathbb{Z}^3$ is bounded by the effective resistance of the tree, and hence it is finite.

This idea has been used to show that random walk on the infinite component of supercritical percolation cluster is transient in dimensions $d \ge 3$ (see Grimmett, Kesten and Zhang (1993)). $\square$

2.4 Spanning trees

Definition 2.24. Let G be a finite connected graph. A spanning tree of G is a subgraph that is a tree (no cycles) and which contains all the vertices of G.

Let G be a finite connected graph and let $\mathcal{T}$ be the set of spanning trees of G. We pick T uniformly at random from $\mathcal{T}$. We call T a uniform spanning tree (UST).

We will prove that T has the property of negative association, i.e.

Theorem 2.25. *Let $G = (V, E)$ be a finite graph. Let $f, g \in E$ with $f \neq g$. Let T be a UST. Then*

$$\mathbb{P}(f \in T \mid g \in T) \leq \mathbb{P}(f \in T).$$

In order to prove this theorem we are first going to establish a connection between spanning trees and electrical networks.

Let $\mathcal{N}(s, a, b, t)$ be the set of spanning trees of G with the property that the unique path from s to t passes along the edge (a, b) in the direction from a to b. We write $N(s, a, b, t) = |\mathcal{N}(s, a, b, t)|$.

Let N be the total number of spanning trees of G. We then have the following theorem:

Theorem 2.26. *The function*

$$i(a, b) = \frac{N(s, a, b, t) - N(s, b, a, t)}{N}$$

for all $(a, b) \in E$ defines a unit flow from s to t satisfying Kirchoff's laws.

Remark 2.27. The above expression for $i(a, b)$ is also equal to

$$i(a, b) = \mathbb{P}(T \in \mathcal{N}(s, a, b, t)) - \mathbb{P}(T \in \mathcal{N}(s, b, a, t)).$$

Exactly the same proof as below would work if G is a weighted graph. In this case we would define the weight of a tree to be

$$w(T) = \prod_{e \in T} c(e)$$

and we would set

$$N^* = \sum_{T \in \mathcal{T}} w(T) \quad \text{and} \quad N^*(s, a, b, t) = \sum_{T \in \mathcal{N}(s,a,b,t)} w(T).$$

Then Theorem 2.26 would still be valid with

$$i^*(a, b) = \frac{\mathcal{N}^*(s, a, b, t) - \mathcal{N}^*(s, b, a, t)}{N^*}$$

for all edges (a, b). The negative association theorem would also be true in this case, i.e. when a tree T is picked with probability proportional to its weight.

Proof of Theorem 2.26. It is obvious from definition that i is an antisymmetric function. We next check that it satisfies Kirchoff's node law, i.e. for all $x \notin \{s,t\}$ we have

$$\sum_{x\sim a} i(a,x) = 0.$$

We now count the contribution of each spanning tree T to the sum above. We now consider the unique path from s to t in this spanning tree. If a is a vertex on this path, then there are two edges on the path with endpoint a that contribute to the sum. The edge going into a and the one going out of a. The first one will contribute $-1/N$ and the second one $1/N$. Now if a is not on the path, then there is no contribution to the sum from T. Hence the overall contribution of T is $-1/N + 1/N = 0$ and this proves Kirchoff's node law.

We now check that it satisfies the cycle law. Let $v_1, \ldots, v_n, v_{n+1} = v_1$ constitute a cycle C. We will show that

$$\sum_{i=1}^{n} i(v_i, v_{i+1}) = 0. \tag{2.2}$$

To do this we will work with *bushes* instead of trees. We define an s/t bush to be a forest consisting of exactly two trees T_s and T_t such that $s \in T_s$ and $t \in T_t$. Let $e = (a,b)$ be an edge. We define $\mathcal{B}(s,a,b,t)$ as the set of s/t bushes with $a \in T_s$ and $b \in T_t$.

We now claim that $|\mathcal{B}(s,a,b,t)| = N(s,a,b,t)$. Indeed, for every bush in $\mathcal{B}(s,a,b,t)$ by adding the edge e we obtain a spanning tree of $\mathcal{N}(s,a,b,t)$. Also for every spanning tree $T \in \mathcal{N}(s,a,b,t)$ by removing the edge e we obtain a bush in $\mathcal{B}(s,a,b,t)$.

So instead of counting the contribution of each spanning tree to the sum in (2.2) we look at bushes. Let B be an s/t bush. Then B makes a contribution to $i(a,b)$ of $1/N$ if $B \in \mathcal{B}(s,a,b,t)$, $-1/N$ if $B \in \mathcal{B}(s,b,a,t)$ and 0 otherwise.

So in total an s/t bush B contributes $(F_+ - F_-)/N$, where F_+ is the number of pairs (v_j, v_{j+1}) so that $B \in \mathcal{B}(s, v_j, v_{j+1}, t)$ and similarly for F_-.

But since C is a cycle, if there is a pair (v_j, v_{j+1}) in F_+, then there must be a pair (v_i, v_{i+1}) in F_-. (If not, this would violate the no cycle property of the tree.) Therefore we get $F_+ = F_-$ and hence the total contribution of B is 0.

Finally we need to check that i is a unit flow, i.e.

$$\sum_{x\sim s} i(s,x) = 1.$$

First we note that $N(s,x,s,t) = 0$ for all $x \sim s$. Every spanning tree must contain a path from s to t, and hence this gives that

$$\sum_{x\sim s} N(s,s,x,t) = N$$

and concludes the proof. □

Proof of Theorem 2.25. We consider G as a network with every edge having conductance 1. Let

$e = (s,t)$ be an edge. Then from Theorem 2.26 we get that i is a unit current flow from s to t and

$$i(s,t) = \frac{N(s,s,t,t)}{N},$$

where $N(s,s,t,t)$ is the number of spanning trees that use the edge (s,t). Hence

$$\frac{N(s,s,t,t)}{N} = \mathbb{P}(e \in T)\,.$$

Since the network has unit conductances, we get that

$$i(s,t) = \varphi(s) - \varphi(t),$$

where φ is the potential associated to the unit current i. Therefore the effective resistance between s and t is given by

$$R_{\text{eff}}(s,t) = i(s,t) = \mathbb{P}(e \in T)\,.$$

Let e and g be distinct edges of G. We write $G.g$ for the graph obtained by gluing both endpoints of g to a single vertex. In this way we obtain a one to one correspondence between spanning trees of G containing g and spanning trees of $G.g$. Therefore, $\mathbb{P}(e \in T \mid g \in T)$ is the proportion of spanning trees of $G.g$ containing e. So from the above

$$\mathbb{P}(f \in T \mid g \in T) = R_{\text{eff}}(s,t;G.g).$$

Gluing the two endpoints of g decreases the effective resistance by Rayleigh's principle, and hence

$$R_{\text{eff}}(s,t;G.g) \le R_{\text{eff}}(s,t;G),$$

which is exactly the statement of the theorem. $\square$

Definition 2.28. Let G be a finite connected graph. We write $\mathcal{F}$ for the set of forests of G (subsets of G that do not contain cycles). Let F be a forest picked uniformly at random among all forests in $\mathcal{F}$. We refer to it as USF.

Conjecture 2.29. *For $f, g \in E$ with $f \neq g$ the USF satisfies*

$$\mathbb{P}(f \in F \mid g \in F) \le \mathbb{P}(f \in F)\,.$$

There is a computer aided proof (Grimmett and Winkler) which shows that for graphs on 8 or fewer vertices this conjecture is true.

Theorem 2.30 (Foster's theorem)**.** *Let G be a finite connected network on n vertices with conductances $(c(e))$ on the edges. Then*

$$\sum_{e \in E} c(e) R_{\text{eff}}(e) = n - 1.$$

Proof. Note that if T is a UST in G, then $\sum_{e \in E} \mathbb{P}(e \in T) = n - 1$. Using that $\mathbb{P}(e \in T) = c(e) R_{\text{eff}}(e)$ concludes the proof. $\square$

2.5 Wilson's algorithm

This section has been taken from Lyons and Peres [6].

In this section we will present a way to sample a uniform spanning tree of a given graph. This algorithm was invented by David Wilson.

In order to describe the algorithm, we start by explaining the loop-erasure of a path. Let $\gamma = \langle x_1, \ldots, x_k \rangle$ be a finite path in a graph which could be directed or undirected. We define its loop erasure by erasing cycles in the chronological order in which they appear. More formally, let $y_0 = x_0$ and if $x_k = x_0$, then we set $m = 0$ and stop here. Otherwise let $i = \max\{j \geq 1 : x_j = x_0\}$ and define $y_1 = x_{i+1}$. If $y_1 = x_k$, then we terminate, otherwise we continue in the same way as before.

Now in order to generate a spanning tree of a graph $G = (V, E)$, we distinguish one vertex which we call the root r and choose any ordering of $V \setminus \{r\}$. We now define a growing sequence of trees by first setting $T_0 = \{r\}$ and then inductively: if T_i spans the whole graph, then we stop, otherwise we pick the first vertex in our ordering which is not in T_i and start a simple random walk on the graph until the first time that it hits T_i. Then we define the tree T_{i+1} by appending to T_i the loop erasure of the random walk path that we generated. This is called Wilson's method.

Theorem 2.31 (Wilson's algorithm)**.** *Wilson's method yields a uniform spanning tree.*

To prove the above miraculous statement we first prove some deterministic results about cycle popping.

Here is another way of thinking about a random walk on the graph. We imagine an infinite stack of cards (or instructions) lying under every vertex. On the i-th visit to the vertex we use the i-th card, which tells us to which vertex we should jump at the next step. Every time we use a card, we throw it away. The instructions on each card are i.i.d. and on vertex x the instructions contain one of the neighbours of x with equal probability.

This is a way of generating a random walk (or more generally any Markov chain), but here we want to sample a uniform spanning tree. So we put an empty stack of cards under the root r and we use the other stacks as follows: at any time the top cards of the stacks give rise to a directed graph whose edges are (x, y) where y is the instruction given in the top card underneath x. We call this the **visible graph** at that time. If it happens that this contains no cycles, then we are done and we have obtained a spanning tree rooted at r. If, however, there is a cycle, then we pop it, which means that we remove the top card in the stacks under the vertices that lie on the cycle. Then we continue with another cycle, until there is no cycle.

We claim that in this way we obtain a spanning tree with probability 1 and it has the uniform distribution. Moreover, as we will show it has the same distribution as the spanning tree obtained by running Wilson's algorithm.

To prove both statements, we will keep track of the locations in the stack of the cards that are popped, and we will call them colours. So we say that an edge (x, S_x^i) has colour i, where S_x^i is the i-th instruction at the vertex x. A coloured cycle is a cycle whose vertices are coloured (but they do not need to have the same colour).

We now state and prove a deterministic statement about popping cycles.

Lemma 2.32. *Given any stacks under the states, the order in which cycles are popped is irrelevant in the sense that every order pops an infinite number of cycles or every order pops the same (finite set of) coloured cycles, thus leaving the same coloured spanning tree on top in the latter case.*

Proof. We will show that if C is any coloured cycle that can be popped, that is, there is some sequence $C_1, C_2, \dots, C_n = C$ that may be popped in that order, but some coloured cycle $C' \neq C_1$ happens to be the first coloured cycle popped, then (1) $C = C'$, or else (2) C can still be popped from the stacks after C' is popped. Once we show this, we are done, since if there are an infinite number of coloured cycles that can be popped, the popping can never stop; whereas in the alternative case, every coloured cycle that can be popped will be popped. Now if all the vertices of C' are disjoint from those of $C_1, C_2, \dots, C_n$, then of course C can still be popped. Otherwise, let C_k be the first cycle that has a vertex in common with C'. Now, all the edges of C' have colour 1. Consider any vertex x in $C' \cap C_k$. Since $x \notin C_1 \cup \dots C_{k-1}$,the edge in C_k leading out of x also has colour 1, so it leads to the same vertex as it does in C'. We can repeat the argument for this successor vertex of x, then for its successor, and so on, until we arrive at the conclusion that $C' = C_k$. Thus, $C' = C$ or we can pop C in the order $C', C_1, C_2, \dots, C_{k1}, C_{k+1}, \dots, C_n$. □

Proof of Theorem 2.31. Wilsons method (using loop-erased parts of a Markov chain) certainly stops with probability 1 at a spanning tree. Using stacks to run the Markov chain and noting that loop erasure in order of cycle creation is one way of popping cycles, we see that Wilsons method pops all the cycles lying over a spanning tree. Because of Lemma 4.2, popping cycles in any other manner also stops with probability 1 and with the same distribution. Furthermore, if we think of the stacks as given in advance, then we see that all our choices inherent in Wilson's method have no effect whatsoever on the resulting spanning tree. Now to show that the distribution is the desired one, think of a given set of stacks as defining a finite set O of coloured cycles lying over a non-coloured spanning tree T . We do not need to keep track of the colours in the spanning tree, since they are easily recovered from the colours in the cycles over it. Let X be the set of all pairs (O, T) that can arise from stacks corresponding to our given Markov chain. If $(O, T) \in X$, then also $(O, T') \in X$ for any other spanning tree T': indeed, anything at all can be in the stacks under any finite set O of coloured cycles. That is, $X = X_1 \times X_2$, where X_1 is a certain collection of sets of coloured cycles and X_2 is the set of all non-coloured spanning trees. Extend our definition of Ψ to coloured cycles C by $\Psi(C) := \prod_{e \in C} p(e)$ and to sets O of coloured cycles by $\Psi(O) := \prod_{C \in O} \Psi(C)$. What is the chance of seeing a given set O of coloured cycles lying over a given spanning tree T? It is simply the probability of seeing all the arrows in $\bigcup O \cup T$ in their respective places, which is simply the product of $p(e)$ for all $e \in \bigcup O \cup T$, in other words, $\Psi(O)\Psi(T)$. Letting $\mathbf{P}$ be the law of (O, T), we get $\mathbf{P} = \mu_1 \times \mu_2$, where μ_i are probability measures proportional to Ψ on X_i. Therefore, the set of coloured cycles seen is independent of the coloured spanning tree and the probability of seeing the tree T is proportional to $\Psi(T)$. This shows that Wilson's method does what we claimed it does. □

Corollary 2.33 (Cayley, 1889)**.** *The number of labeled unrooted trees with n vertices is n^{n-2}. Here, a labeled tree is one whose vertices are labeled 1 through n.*

We now mention another way of sampling uniform spanning trees due to Aldous and Broder. Their algorithm proceeds as follows: choose a vertex r of G and start a simple random walk until the first time that all vertices have been visited at least once. Then for every $w \in V$ with $w \neq r$ let (u, w) be the directed edge that was traversed by the walk on its first visit to w. The edges obtained in this way undirected constitute a uniform spanning tree.

2.6 Uniform spanning trees and forests

In the previous section we described an algorithm to generate a uniform spanning tree of a finite connected graph. We will now see how to define a measure on spanning trees on infinite recurrent graphs and uniform spanning forests on transient graphs.

We will do this by taking limits over exhaustions of the infinite graph by finite subgraphs.

Let us first look at the recurrent case. Let G be an infinite recurrent graph and let r be a distinguished vertex that will be the root of our spanning tree. We now start a random walk from some other vertex of G and run it until it first hits r. Then we take its loop erasure. We now start another walk from a point not already taken and start a random walk until the first time it hits the existing tree and we apply loop erasure. We continue in the same way. In this way we obtain a spanning tree of the infinite graph. The reason we call it a uniform spanning tree is because if we restrict it to a big finite subgraph G' of G, then the probabilities that a uniform spanning tree of G' contains a finite set of edges is close to the probability that a uniform spanning tree of G contains the same set of edges.

In the case of a transient graph we start by defining $\mathcal{F}_0 = \varnothing$. Then inductively to define $\mathcal{F}_{i+1}$ we run a random walk from a point not already taken and we stop the random walk if it hits $\mathcal{F}_i$ or if not, we let it run indefinitely. Then take its loop erasure, which is well defined since the walk is transient, so it visits every state only finitely many times. Then we append the loop erasure to the previous set and we continue in the same way. It is clear that what we get will contain no cycles. This method of generating a uniform spanning forest was called by Benjamini, Lyons, Pemantle and Peres (2001) Wilson's method rooted at infinity.

Let (G_n) be an exhaustion of an infinite graph G. Let μ_n be the uniform spanning tree measure on G_n where we glue all the vertices of $G \setminus G_n$ into a single vertex. The spanning forest we obtain in the limit is also called wired uniform spanning forest as opposed to the free spanning forest where there is no gluing taking place.

Theorem 2.34. *The weak limit $\mu = \lim_{n\to\infty} \mu_n$ exists and is supported on the set of spanning forests of G. Moreover, the measure μ is the distribution of the spanning forest generated using Wilson's method rooted at infinity.*

For a proof see [4, Theorem 2.11] and Lyons and Peres (Chapter 10).

We now turn to see in which cases the uniform spanning forest consists of a single tree. From the construction it is clear that for a recurrent graph, this is always the case by construction. For transient graphs the answer is in the next theorem.

Pemantle first answered this in 1991 and it was quite striking.

Proposition 2.35 (Pemantle (1991))**.** *The uniform spanning forest is almost surely a tree if and only if starting from every vertex a simple random walk and an independent loop-erased random walk intersect infinitely many times with probability* 1*. Moreover, the probability that x and y are in the same tree is equal to the probability that simple random walk started from x intersects an independent loop-erased random walk started from y.*

This proposition is obvious using Wilson's method, but this method of generating uniform spanning forests was not known to Pemantle.

In 2003 Lyons, Peres and Schramm showed that a random walk intersects an independent loop-erased walk infinitely often if and only if the same is true for two independent random walks. So then we get the following:

Theorem 2.36. *Let G be an infinite transient graph. Then the uniform spanning forest is a single tree if and only if two random walks started at any two vertices intersect with probability 1.*

Let us now apply the previous theorem to spanning forests on $\mathbb{Z}^d$.

Theorem 2.37. *The uniform spanning forest in $\mathbb{Z}^d$ is a single tree almost surely for all $d \leq 4$.*

Proof *for $d \geq 5$.* Let X and Y be two independent simple random walks on $\mathbb{Z}^d$ starting from $X_0 = x$ and $Y_0 = y$.

We start by calculating the expected number of intersections

$$I = \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \mathbf{1}(X_t = Y_s).$$

For this we have

$$\mathbb{E}_{x,y}[I] = \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \mathbb{P}_{x,y}(X_t = Y_s) = \sum_{t,s} \mathbb{P}_{x-y}(X_{t+s} = 0) = \sum_{t=0}^{\infty} (t+1) \cdot \mathbb{P}_{x-y}(X_t = 0)\,.$$

Since simple random walk only moves distance 1 at each time step, the sum above should start from $\|x - y\|$. Then using that there exists a positive constant c so that for all x we have

$$\mathbb{P}_x(X_t) \leq \frac{c}{t^{d/2}}$$

we get that

$$\mathbb{E}_{x,y}[I] \leq \sum_{t=\|x-y\|}^{\infty} \frac{c}{t^{d/2-1}}.$$

Now for $\|x - y\|$ sufficiently large and $d \geq 5$ we see that this last sum can be made smaller than ε. Therefore, we get by Markov's inequality that

$$\mathbb{P}_{x,y}(I > 0) \leq \mathbb{E}_{x,y}[I] < \varepsilon.$$

For every $\varepsilon > 0$ we can find x, y so that $\mathbb{E}_{x,y}[I] < \varepsilon$. Therefore, from Wilson's method rooted at infinity we obtain

$$\mathbb{P}(\text{USF is connected}) \leq \mathbb{P}_{x,y}(X \text{ and } Y \text{ intersect}) \leq \varepsilon.$$

Since this is true for all $\varepsilon > 0$, this gives that with probability 1 the uniform spanning forest consists of a single tree. □

Let us now prove that for simple random walk on $\mathbb{Z}^d$ for all x and all t we have

$$\mathbb{P}_x(X_t = 0) \leq \frac{c}{t^{d/2}}.$$

It is easy to check by direct calculation and using Stirling's formula that for a simple random walk Y on $\mathbb{Z}$ we have

$$\mathbb{P}_x(Y_t = 0) \leq \frac{c}{\sqrt{t}},$$

where c is a positive constant. Now if we write $X_t = (X_t^1, \ldots, X_t^d)$ and denote by t_i the number of steps that the walk X^i performed up to time t, then we get

$$\mathbb{P}_x(X_t = 0) \leq \mathbb{P}_x\left(X_t = 0, t_i > \frac{t}{2d}, \forall\, i\right) + \mathbb{P}\left(\exists i :\, t_i < \frac{t}{2d}\right) \leq \frac{c_1}{t^{d/2}} + e^{-c_2 t}.$$

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